Supplementary 9: Linear Algebra Essentials for Modeling
Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this two-hour supplementary lesson, students should be able to:

- represent data, decisions, and states using **vectors**;
- compute and interpret **dot products**, norms, linear combinations, and span;
- read a **matrix** as a table, a linear transformation, or a system of equations;
- solve small systems by **Gaussian elimination**;
- interpret **rank**, free variables, and the solution structure of homogeneous systems;
- connect linear algebra to least squares, Markov chains, game theory, and dimensional analysis;
- explain why eigenvectors and stationary distributions matter in dynamic modeling.

Why linear algebra belongs in modeling

Core explanation, worked reasoning, and application

Why linear algebra belongs in modeling

Calculus describes how one quantity changes. Linear algebra describes how many quantities combine. In mathematical modeling, this is unavoidable: a delivery route has many travel times, a regression model has many residuals, a Markov chain has many transition probabilities, and a dimensional-analysis problem has many physical exponents.

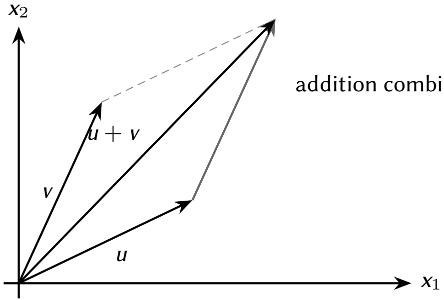
Activity: Launch activity: where are the vectors?

For each modeling situation, identify a natural vector.

1. A robot route visits five buildings and records five travel times.
2. A school canteen records arrivals in six ten-minute intervals.
3. A company tracks market share among three brands over many weeks.
4. A regression model compares predicted and observed values for ten data points.

The goal is not to make notation heavier. It is to make multi-variable information readable and computable.

Illustration: Why linear algebra belongs in modeling



addition combines effects

Vector addition models combined effects, such as displacement, resource use, or accumulated changes across components.

Vectors as data packages

Core explanation, worked reasoning, and application

Concept: Vectors, dot products, and norms

A vector in $\mathbb{R}^n$ is an ordered list

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

For $u, v \in \mathbb{R}^n$,

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n, \qquad \|v\| = \sqrt{v \cdot v}.$$

The dot product measures alignment. If $u \cdot v = 0$, the vectors are **orthogonal**.

Worked Example: Worked example 1: resource-use vector

A delivery robot uses battery, time, and staff attention. For one route, suppose

$$x = \begin{pmatrix} 18 \\ 42 \\ 3 \end{pmatrix}$$

means 18 battery units, 42 minutes, and 3 staff interventions. A cost vector

$$c = \begin{pmatrix} 0.4 \\ 1.2 \\ 8 \end{pmatrix}$$

assigns weights to each component. The total weighted cost is

$$c \cdot x = 0.4(18) + 1.2(42) + 8(3) = 81.6.$$

This is the same idea as an LP objective $c \cdot x$ in Lecture 6.

Concept: Linear combination and span

A **linear combination** of $v_1, \ldots, v_k$ is

$$c_1 v_1 + \cdots + c_k v_k.$$

The set of all such combinations is the **span**, written

$$\mathrm{span}(v_1, \ldots, v_k).$$

Vectors are **linearly independent** if the only way to make $\mathbf{0}$ is to use all zero coefficients.

Worked Example: Worked example 2: detecting dependence

Let

$$v_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\4\\2\end{pmatrix}.$$

Since

$$v_1 + 2v_2 = \begin{pmatrix}1\\2\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix} = v_3,$$

the three vectors are linearly dependent. In modeling terms, the third feature adds no new direction; it is already explained by the first two.

Checkpoint: Checkpoint

In $\mathbb{R}^3$, can four vectors be linearly independent? Explain without computation.

Matrices as structured models

Core explanation, worked reasoning, and application

Concept: Matrix multiplication

An $m \times n$ matrix $\mathbf{A}$ maps vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ by

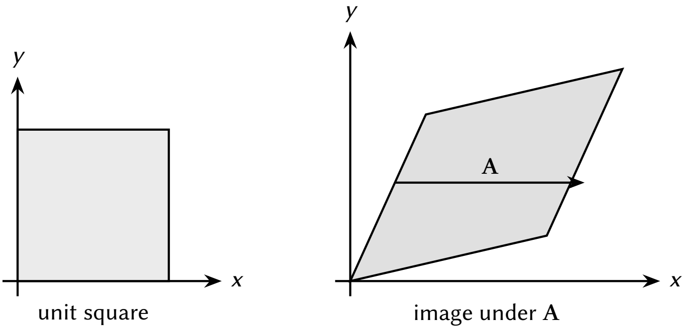
$$y = \mathbf{A}x.$$

If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$, then

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

The product $\mathbf{AB}$ represents composition: do $\mathbf{B}$ first, then $\mathbf{A}$.

Illustration: Matrices as structured models



A matrix can be viewed as a transformation. The columns of $\mathbf{A}$ show where the unit basis vectors move.

Worked Example: Worked example 3: matrix product as a transition

Suppose a two-state weather model uses the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix},$$

where rows represent today's state and columns represent tomorrow's state: sunny S or rainy R. If today is certainly sunny, the row vector is

$$x_0^\top = (1, 0).$$

After one day,

$$x_1^\top = x_0^\top \mathbf{P} = (0.8, 0.2).$$

After two days,

$$x_2^\top = x_0^\top \mathbf{P}^2 = (0.72, 0.28).$$

Matrix powers describe repeated transitions, which is the core of Markov-chain modeling.

Model Critique: Model critique: what does a matrix assume?

A transition matrix assumes that the next state depends on the current state in a fixed way. For weather, customer movement, or disease progression, ask:

- Are transition probabilities stable over time?
- Does tomorrow depend only on today, or also on the previous week?
- Are all relevant states included?

A matrix is compact, but compact notation can hide strong assumptions.

Solving systems by Gaussian elimination

Core explanation, worked reasoning, and application

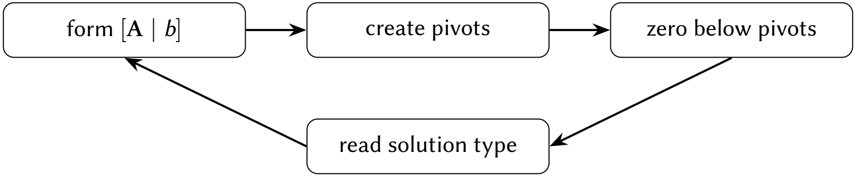
Concept: Linear system

A system of m linear equations in n unknowns can be written as

$$\mathbf{A}x = b.$$

Gaussian elimination uses row operations on the augmented matrix $[\mathbf{A} \mid b]$ to reveal pivots, free variables, and consistency.

Summary: Gaussian elimination route



Worked Example: Worked example 4: a 3×3 system

Solve

$$\begin{cases} x + 2y + z = 6, \\ 2x + 3y - z = 4, \\ -x + y + 4z = 9. \end{cases}$$

The augmented matrix reduces as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 3 & -1 & 4 \\ -1 & 1 & 4 & 9 \end{array}\right] \xrightarrow{R_2 - 2R_1,\ R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -1 & -3 & -8 \\ 0 & 3 & 5 & 15 \end{array}\right]$$

$$\xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & -4 & -9 \end{array}\right].$$

Back substitution gives $z = 9/4$, $y = 5/4$, and $x = 5/4$. Therefore

$$(x, y, z) = \left(\frac{5}{4}, \frac{5}{4}, \frac{9}{4}\right).$$

Concept: Rank and solution type

The **rank** of a matrix is the number of pivots in row echelon form. A system has:

- no solution if row reduction gives $[0\ 0\ \cdots\ 0 \mid c]$ with $c \neq 0$;
- exactly one solution if every variable column has a pivot;
- infinitely many solutions if the system is consistent but has free variables.

For $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\dim(\text{null space}) = n - \operatorname{rank}(\mathbf{A}).$$

Worked Example: Worked example 5: null space and dimensionless products

For a pendulum model, suppose the period T may depend on length L, mass m, and gravitational acceleration g. A dimensionless product has the form

$$T^a L^b m^c g^d.$$

Using dimensions $[T] = (0,0,1)$, $[L] = (0,1,0)$, $[m] = (1,0,0)$, and $[g] = (0,1,-2)$ in rows M, L, T, the dimension equations are

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}.$$

The equations are $c = 0$, $b + d = 0$, and $a - 2d = 0$. Taking $d = 1$ gives

$$(a, b, c, d) = (2, -1, 0, 1),$$

so one dimensionless product is

$$\Pi = \frac{T^2 g}{L}.$$

This is exactly the null-space calculation behind Buckingham's Π theorem.

Least squares as linear algebra

Core explanation, worked reasoning, and application

Concept: Residual vector and normal equations

For data (x_i, y_i) and a line $y = a + bx$, define

$$\mathbf{X} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

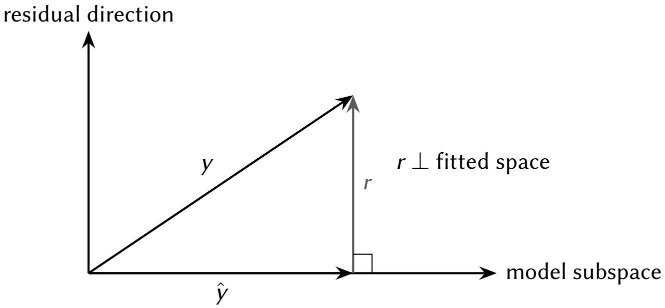
Predictions are $\hat{y} = \mathbf{X}\boldsymbol{\beta}$ and residuals are

$$r = y - \mathbf{X}\boldsymbol{\beta}.$$

Least squares chooses $\boldsymbol{\beta}$ to minimise $\|r\|^2$. The normal equations are

$$\mathbf{X}^\top \mathbf{X}\boldsymbol{\beta} = \mathbf{X}^\top y.$$

Illustration: Least squares as linear algebra



Least squares chooses the fitted vector $\hat{y}$ so that the residual vector is orthogonal to the model subspace. This gives $\mathbf{X}^\top r = \mathbf{0}$.

Worked Example: Worked example 6: fitting a line

Fit $y = a + bx$ to the data

$$(0,1),\ (1,2),\ (2,2),\ (3,4).$$

Then

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}.$$

Compute

$$\mathbf{X}^\top \mathbf{X} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}, \qquad \mathbf{X}^\top y = \begin{pmatrix} 9 \\ 18 \end{pmatrix}.$$

Solve

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}.$$

The solution is $a = 0.9$, $b = 0.9$. The fitted model is

$$\hat{y} = 0.9 + 0.9x.$$

Model Critique: Model critique: matrix calculation is not the whole model

The normal equations compute the best line under a squared-error criterion. A modeling report still needs to discuss residual patterns, units, outliers, whether the relationship should be linear, and whether the prediction range is appropriate.

Eigenvalues, stationary states, and long-run behavior

Core explanation, worked reasoning, and application

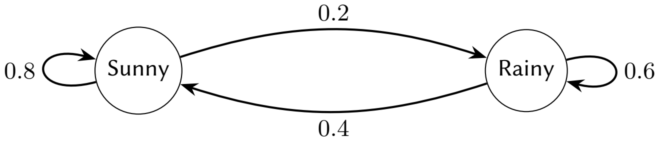
Concept: Eigenvalue and eigenvector

For a square matrix $\mathbf{A}$, a non-zero vector v is an **eigenvector** with **eigenvalue** λ if

$$\mathbf{A}v = \lambda v.$$

The direction of v is preserved by the transformation; only its scale changes.

Illustration: Eigenvalues, stationary states, and long-run behavior



A two-state Markov chain. The stationary distribution is a left eigenvector of the transition matrix with eigenvalue 1.

Worked Example: Worked example 7: stationary distribution

For

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix},$$

a stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2)$ satisfies

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}, \qquad \pi_1 + \pi_2 = 1.$$

The first equation gives

$$0.8\pi_1 + 0.4\pi_2 = \pi_1 \quad \Rightarrow \quad 0.4\pi_2 = 0.2\pi_1 \quad \Rightarrow \quad \pi_1 = 2\pi_2.$$

Together with $\pi_1 + \pi_2 = 1$, we get

$$\boldsymbol{\pi} = \left(\frac{2}{3}, \frac{1}{3}\right).$$

In the long run, about two thirds of days are sunny in this model.

Worked Example: Worked example 8: population matrix

Let J_n and A_n be juvenile and adult populations in year n. Suppose each adult produces 0.6 juveniles, 30% of juveniles mature, and 80% of adults survive. Then

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0.6 \\ 0.3 & 0.8 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} J_n \\ A_n \end{pmatrix}.$$

The dominant eigenvalue of $\mathbf{M}$ describes the long-run growth factor. Solving

$$\det(\mathbf{M} - \lambda \mathbf{I}) = \det \begin{pmatrix} -\lambda & 0.6 \\ 0.3 & 0.8 - \lambda \end{pmatrix} = \lambda^2 - 0.8\lambda - 0.18 = 0$$

gives $\lambda \approx 0.983$. The model predicts a slow long-run decline, because the dominant growth factor is slightly below 1.

Differentiated modeling studio

Core explanation, worked reasoning, and application

Activity: Modeling studio

Choose one route according to readiness.

Core route. Solve a 3×3 linear system by elimination and explain the solution type.

Challenge route. Fit a line to five data points using the normal equations and compute the residual vector.

Extension route. Find the stationary distribution of a three-state transition matrix and interpret the long-run state proportions.

Each group should write one short paragraph explaining what the matrix represents before doing the computation.

Summary: Competition-style communication

A concise modeling report paragraph might read:

We represented the system state as a vector and encoded transitions by a matrix. Repeated behavior was computed using matrix multiplication, while the long-run distribution was obtained by solving the stationary equation $\boldsymbol{\pi} P = \boldsymbol{\pi}$ *with* $\sum_i \pi_i = 1$. *This interpretation assumes transition probabilities are stable and that the current state contains enough information to predict the next step.*

Checkpoint: Exit ticket

Answer in three sentences:

1. What does a pivot tell you in Gaussian elimination?
2. Why is a residual vector useful in least squares?
3. What does a stationary distribution mean in a Markov-chain model?

Exercises

Core explanation, worked reasoning, and application

Exercises: Exercise 1: Vectors

1. Find a unit vector in the direction of $(3, 4, 0)$.
2. Compute the dot product of $(1, 2, 3)$ and $(4, -1, 2)$, then find the angle between them.
3. Determine whether $(1, 0, 2)$, $(3, 1, 4)$, and $(2, 1, 0)$ are linearly independent.
4. In a delivery model, $x = (8, 20, 2)$ records distance, time, and number of delays. If $c = (1.5, 0.8, 10)$, interpret $c \cdot x$.

1. Compute **AB** and **BA** for

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}.$$

2. Solve by Gaussian elimination:

$$\begin{cases} 2x + y - z = 3, \\ x - y + 2z = 1, \\ 3x + 2y + z = 10. \end{cases}$$

3. Explain why the system

$$x + 2y + 3z = 1, \qquad 2x + 4y + 6z = 3$$

   has no solution.

4. Find all solutions of

$$\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 0, \\ 2x_1 - x_2 + x_3 + x_4 = 0. \end{cases}$$

1. Fit $y = a + bx$ to $(1,2), (2,3), (3,5), (4,4), (5,6)$ using $\mathbf{X}^\top\mathbf{X}\boldsymbol{\beta} = \mathbf{X}^\top y$.
2. For
$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix},$$
find the stationary distribution.
3. A zero-sum game has payoff matrix
$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}.$$
Find Row's mixed strategy by making Column indifferent.
4. In dimensional analysis, explain why the number of independent dimensionless groups equals "number of variables minus rank of the dimension matrix."

1. Find eigenvalues and eigenvectors of $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
2. Let $\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Explain geometrically why $\mathbf{R}(\alpha)\mathbf{R}(\beta) = \mathbf{R}(\alpha + \beta)$.
3. Prove that if v is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then v is an eigenvector of $\mathbf{A}^k$ with eigenvalue λ^k.
4. For a square matrix, list three equivalent ways to say that it is invertible.

Exercises

Appendix A: Algorithm summary

Core explanation, worked reasoning, and application

Appendix A: Algorithm summary

Task	**Practical method**
Solve $\mathbf{A}x = b$	Row-reduce $[\mathbf{A} \mid b]$, then back-substitute.
Find a null space	Row-reduce $\mathbf{A}$, assign parameters to free variables, write basis vectors.
Test independence	Put vectors as columns; check whether every column has a pivot.
Find least-squares line	Form $\mathbf{X}$, compute $\mathbf{X}^\top\mathbf{X}$ and $\mathbf{X}^\top y$, solve normal equations.
Find stationary distribution	Solve $\boldsymbol{\pi} P = \boldsymbol{\pi}$ plus $\sum_i \pi_i = 1$.
Find eigenvalues	Solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
Find eigenvectors	For each eigenvalue, solve $(\mathbf{A} - \lambda\mathbf{I})v = \mathbf{0}$.

Appendix B: Connection map to the main lectures

Core explanation, worked reasoning, and application

Appendix B: Connection map to the main lectures

Tool	**Where it appears**
Dot product and norm	Lecture 3 residual vector; Lecture 6 linear objective $c \cdot x$.
Matrix multiplication	Lecture 5 transition matrices; Lecture 8 mixed strategies.
Gaussian elimination	Lecture 6 simplex thinking; Lecture 11 dimensionless products.
Rank and null space	Lecture 11 Buckingham Π theorem; Lecture 3 design matrix.
Least squares	Lecture 3 model fitting and residual analysis.
Eigenvalues	Lecture 5 Markov chains; Lecture 9 linear dynamical systems; Lecture 12 discrete systems.