# Supplementary 7: Definite Integrals, Accumulation, and Numerical Integration

Mathematical Modeling Lecture

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# Session Route

### Launch

**Question**

Motivating context, key vocabulary, and the first modelling decision.

### Build

**Method**

Worked example, diagram or table, calculation, and parameter meaning.

### Test

**Critique**

Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

# Learning Goals

By the end of this two-hour supplementary lesson, students should be able to:

- interpret a **definite integral** as signed area, accumulated change, and average value;
- construct and read a **Riemann sum** using left, right, and midpoint rectangles;
- use the **Fundamental Theorem of Calculus** to evaluate basic definite integrals;
- apply **substitution** and **integration by parts** in simple modeling cases;
- recognise when an integral is **improper** and whether it converges;
- compare rectangle, trapezoidal, Simpson, and Monte Carlo thinking as numerical integration tools.

# From area to accumulated change

Core explanation, worked reasoning, and application

# From area to accumulated change

A definite integral is not just a way to compute geometric area. In modeling it often represents an accumulated quantity: total water entering a tank, total cost over time, average inventory, probability mass, or total pollutant load. The same notation

$$\int_a^b f(x)\,dx$$

may mean area, total change, or average effect depending on what $f$ represents.

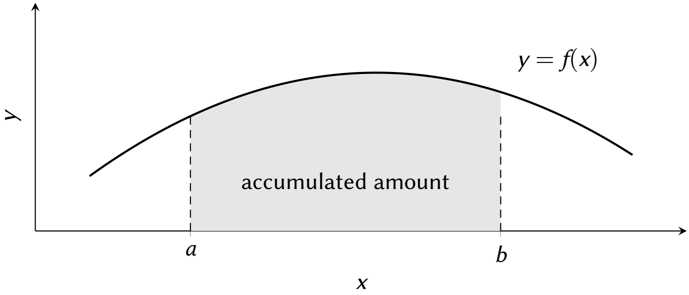
# Activity: Launch activity: what does the integral measure?

Suppose $r(t)$ is the arrival rate of students at a cafeteria, measured in students per minute. What does

$$\int_0^{20} r(t)\,dt$$

represent? It is not a rate. It is the total number of students who arrive during the first 20 minutes.

# Illustration: From area to accumulated change




The definite integral accumulates the vertical quantity $f(x)$ across an interval. If $f$ is a rate, the integral is total change. If $f$ is a density, the integral is total mass.

# Riemann sums: building the integral

Core explanation, worked reasoning, and application

# Concept: Partition and Riemann sum

A **partition** of $[a, b]$ is a list

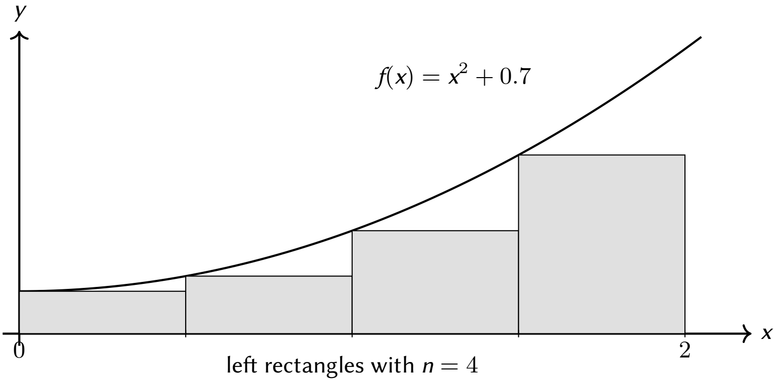
$$a = x_0 < x_1 < \cdots < x_n = b.$$

The width of the $k$-th subinterval is $\Delta x_k = x_k - x_{k-1}$. If $x_k^*$ is a sample point in the subinterval, the corresponding **Riemann sum** is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

For a regular partition, $\Delta x = (b - a)/n$. Common choices are left endpoint, right endpoint, and midpoint.

# Illustration: Riemann sums: building the integral




A left Riemann sum uses the height at the left endpoint of each subinterval. For an increasing function, it underestimates the area.

# Worked Example: Worked example 1: a left Riemann sum

Approximate

$$\int_0^2 (x^2 + 1)\, dx$$

using a left Riemann sum with $n = 4$ equal subintervals.
Here $\Delta x = 0.5$, and the left sample points are $0, 0.5, 1, 1.5$. Thus

$$S_L = 0.5\,[f(0) + f(0.5) + f(1) + f(1.5)] = 0.5(1 + 1.25 + 2 + 3.25) = 3.75.$$

The exact value will later be $14/3 \approx 4.667$, so the left sum underestimates.

# Concept: The definite integral

If the Riemann sums approach a unique number as the mesh of the partition tends to zero, we define

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

When $f \geq 0$, this is area. When $f$ changes sign, it is **signed area**: area above the $x$-axis contributes positively, and area below contributes negatively.

# Checkpoint: Checkpoint

For an increasing positive function on $[a, b]$, which is usually larger: the left sum or the right sum? Explain using rectangles, not formulas.

# The Fundamental Theorem of Calculus

Core explanation, worked reasoning, and application

# The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the two ideas from Supplementary 5 and this lesson: derivative and integral.

# Concept: Fundamental Theorem of Calculus

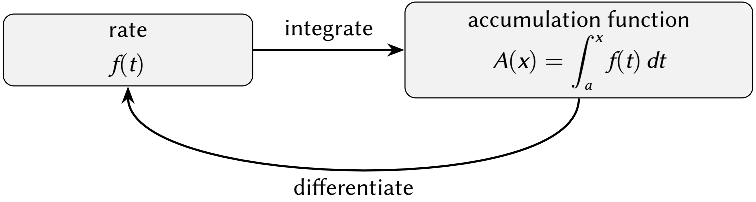
Let $f$ be continuous on $[a, b]$.

- **Part I:** If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.
- **Part II:** If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The first part says accumulation has rate $f(x)$; the second part says definite integrals can be evaluated by antiderivatives.

# Illustration: The Fundamental Theorem of Calculus




$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

FTC Part I says that integration and differentiation undo each other in the accumulation-function setting.

# Worked Example: Worked example 2: evaluating by the FTC

Evaluate

$$\int_0^2 (x^2 + 1)\, dx.$$

An antiderivative is $F(x) = x^3/3 + x$. Hence

$$\int_0^2 (x^2 + 1)\, dx = \left[\frac{x^3}{3} + x\right]_0^2 = \frac{8}{3} + 2 = \frac{14}{3}.$$

This explains why the $n = 4$ left sum $3.75$ was only an approximation.

# Worked Example: Worked example 3: differentiating an integral

Compute

$$\frac{d}{dx}\left(\int_2^{x^2} \sin(t^2)\,dt\right).$$

Let $u = x^2$. By FTC Part I and the chain rule,

$$\frac{d}{dx}\int_2^{u} \sin(t^2)\,dt = \sin(u^2)\frac{du}{dx} = 2x\sin(x^4).$$

The integrand need not have an elementary antiderivative; the derivative is still easy.

# Substitution and integration by parts

Core explanation, worked reasoning, and application

# Concept: Substitution as reverse chain rule

If $u = g(x)$ and $du = g'(x)\,dx$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

For a definite integral, change the limits: $x = a \mapsto u = g(a)$ and $x = b \mapsto u = g(b)$.

# Worked Example: Worked example 4: substitution with definite limits

Evaluate

$$\int_0^2 xe^{x^2}\,dx.$$

Let $u = x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. The limits become $0 \mapsto 0$ and $2 \mapsto 4$. Therefore

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4 - 1).$$

# Concept: Integration by parts as reverse product rule

From $(uv)' = u'v + uv'$, we obtain

$$\int u\,dv = uv - \int v\,du.$$

A useful choice rule is LIATE: logarithmic, inverse-trigonometric, algebraic, trigonometric, exponential. The function earlier in the list is often a good choice for $u$.

# Worked Example: Worked example 5: integration by parts

Evaluate

$$\int_0^1 xe^x \, dx.$$

Choose $u = x$ and $dv = e^x \, dx$. Then $du = \, dx$ and $v = e^x$. Thus

$$\int_0^1 xe^x \, dx = [xe^x]_0^1 - \int_0^1 e^x \, dx = e - (e - 1) = 1.$$

# Checkpoint: Checkpoint

In substitution, the main question is “what inner function has its derivative nearby?” In integration by parts, the main question is “which part becomes simpler after differentiating?”

# Modeling applications of definite integrals

Core explanation, worked reasoning, and application

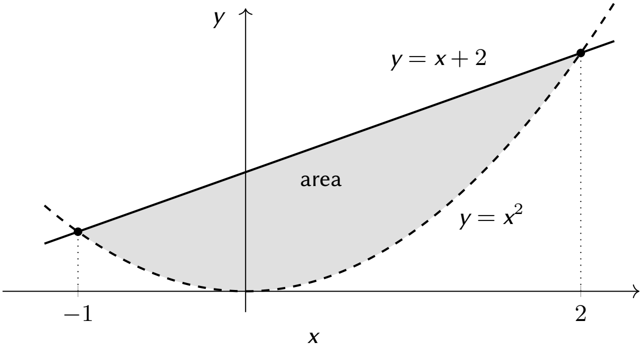
# Concept: Area between two curves

If $f(x) \geq g(x)$ on $[a, b]$, the area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

The order matters: top curve minus bottom curve.

# Illustration: Modeling applications of definite integrals




$$\text{Area} = \int_{-1}^{2} \left[(x+2) - x^2\right] dx$$

For area between curves, integrate upper minus lower. The shaded region is bounded by $y = x + 2$ and $y = x^2$ from $x = -1$ to $x = 2$.

# Worked Example: Worked example 6: area between curves

Find the area enclosed by $y = x^2$ and $y = x + 2$.
Intersections satisfy $x^2 = x + 2$, so $x = -1$ or $x = 2$. On this interval, $x + 2 \geq x^2$. Hence

$$A = \int_{-1}^{2} [(x+2) - x^2]\, dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$

# Concept: Average value and accumulated change

For a continuous function $f$ on $[a, b]$, its **average value** is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

If $Q'(t) = r(t)$, then

$$Q(b) - Q(a) = \int_a^b r(t)\, dt.$$

This is the form used in flow, pollution, population, inventory, and energy models.

# Worked Example: Worked example 7: average inventory

In the EOQ model, suppose inventory decreases linearly from $Q$ to $0$ over a cycle of length $T$:

$$I(t) = Q\left(1 - \frac{t}{T}\right), \qquad 0 \le t \le T.$$

The average inventory is

$$\bar{I} = \frac{1}{T}\int_0^T Q\left(1 - \frac{t}{T}\right)\, dt = \frac{Q}{T}\left[t - \frac{t^2}{2T}\right]_0^T = \frac{Q}{2}.$$

This familiar result is an integral statement, not only a geometric triangle argument.

# Model Critique: Model criticism: units and sign

Before reporting an integral, check its units. If $r(t)$ is litres/hour and $t$ is measured in hours, then $\int r(t)\,dt$ is litres. Also check whether signed area makes sense: negative values may represent outflow, loss, or correction rather than literal negative area.

# Improper integrals and convergence

Core explanation, worked reasoning, and application

# Improper integrals and convergence

Some modeling integrals extend over an infinite interval or include an unbounded integrand. These are **improper integrals**. They must be defined using limits.

# Concept: Two common improper integrals

For an infinite interval,

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

For a vertical blow-up at $a$,

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

If the limit exists, the integral **converges**. Otherwise it **diverges**.

# Worked Example: Worked example 8: convergence versus divergence

Compare

$$\int_1^\infty \frac{1}{x^2}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x}\,dx.$$

For the first,

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1,$$

so it converges. For the second,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty,$$

so it diverges. Infinite interval does not automatically mean infinite area.

# Checkpoint: Checkpoint

Why is the computation $\int_{-1}^{1} 1/x^2\,dx = [-1/x]_{-1}^{1}$ invalid? Identify the hidden discontinuity.

# Numerical integration

Core explanation, worked reasoning, and application

# Numerical integration

Many integrals in modeling do not have elementary antiderivatives. For example, $\int e^{-x^2}\,dx$ appears in statistics, but cannot be expressed using elementary functions. We then approximate.

# Concept: Trapezoidal and Simpson rules

For a regular partition of $[a, b]$ with $n$ subintervals and $h = (b - a)/n$:

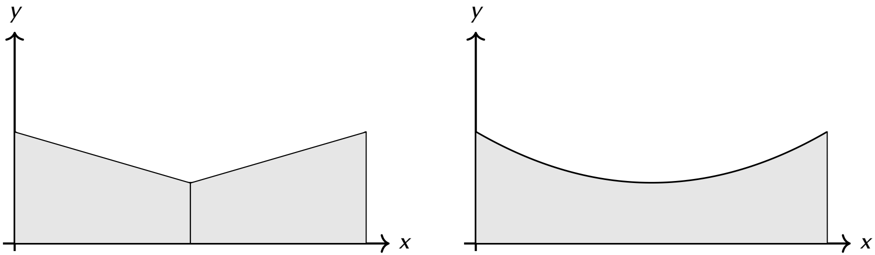
$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

If $n$ is even, Simpson's rule is

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\right].$$

Trapezoidal uses straight-line panels; Simpson uses quadratic panels.

# Illustration: Numerical integration




trapezoidal panels

quadratic panel

Trapezoidal panels approximate the curve by straight segments. Simpson's rule approximates over pairs of subintervals by a quadratic curve.

# Worked Example: Worked example 9: comparing numerical rules

Estimate

$$\int_0^1 e^{-x^2}\,dx$$

using $n = 4$ subintervals. With $h = 0.25$, the function values are approximately

| $x_k$ | $0$ | $0.25$ | $0.50$ | $0.75$ | $1.00$ |
|---|---|---|---|---|---|
| $e^{-x_k^2}$ | $1.00000$ | $0.93941$ | $0.77880$ | $0.56978$ | $0.36788$ |

Thus

$$T_4 = 0.74298, \qquad S_4 = 0.74685.$$

The true value is approximately $0.746824$, so Simpson's rule is much more accurate here.

# Model Critique: Numerical integration in a modeling report

A good report should not only state "we used Simpson's rule." It should specify the interval, step size, stopping criterion, and a validation check. For example:

> *We estimated the integral using Simpson's rule with step size $h = 0.01$. Doubling the number of subintervals changed the value by less than $10^{-4}$, so numerical error is small relative to data uncertainty.*

# Summary and practice

Core explanation, worked reasoning, and application

# Summary: Key ideas

- Riemann sums approximate integrals by finite pieces.
- The definite integral is an accumulation operator: area, total change, average value, probability, or mass.
- FTC connects accumulated change and instantaneous rate.
- Substitution is reverse chain rule; integration by parts is reverse product rule.
- Improper integrals require limits and may converge or diverge.
- Numerical integration is a modeling choice; accuracy and cost should be justified.

# Exercises: Core practice

1. Compute left, right, and midpoint sums for $\int_0^2 x^2\,dx$ with $n = 4$. Compare with the exact value.
2. Evaluate $\int_1^4 (3x^2 - 2x + 5)\,dx$ using the FTC.
3. Evaluate $\int_0^1 2x(1 + x^2)^5\,dx$ using substitution.
4. Evaluate $\int_1^e \ln x\,dx$ using integration by parts.
5. Determine whether $\int_1^\infty x^{-3/2}\,dx$ converges.
6. Find the average value of $f(x) = x^2$ on $[0, 3]$.

# Exercises: Modeling challenge: cafeteria inflow

During the first 30 minutes of lunch, the arrival rate of students is modeled by

$$r(t) = 18 + 0.9t - 0.03t^2, \qquad 0 \le t \le 30,$$

where $t$ is measured in minutes and $r(t)$ in students per minute.

1. Compute the total number of students arriving in the first 30 minutes.
2. Find the average arrival rate.
3. Explain one assumption behind using a smooth rate function for student arrivals.
4. Suggest one way to validate the model using observed queue data.

# Exercises: Extension practice

1. Show that if $f$ is odd and integrable on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$.
2. Use Simpson's rule with $n = 4$ to approximate $\int_0^1 \frac{1}{1+x^2}\,dx$ and compare with $\pi/4$.
3. Verify that $\int_0^\infty \lambda e^{-\lambda x}\,dx = 1$ for $\lambda > 0$, and explain why this matters for probability density models.