Supplementary 6: Applications of Derivatives and Optimization

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this two-hour supplementary lesson, students should be able to:

- use the sign of f' to determine **increasing/decreasing behaviour**;
- locate and classify **critical points**;
- use f'' to discuss **concavity** and **inflection points**;
- sketch a curve from domain, asymptotes, derivative sign, and concavity;
- formulate and solve a one-variable **optimization** model;
- communicate derivative-based conclusions as modeling evidence, not just computation.

Derivative evidence for optimization and model design

Core explanation, worked reasoning, and application

Derivative evidence for optimization and model design

In a modeling report, a derivative is rarely an end in itself. It is evidence for a statement such as:

> *"The waiting time increases slowly at first, then rises sharply near capacity."*

The first derivative tells us the *direction and rate of change*; the second derivative tells us whether the change is accelerating or slowing down. These ideas turn calculus into a practical language for decisions: when to stop production, where a maximum occurs, whether a policy has diminishing returns, and whether a model becomes unstable near a boundary.

Activity: Launch activity: reading a derivative without calculating

A robot delivery system has travel-time function $T(q)$, where q is the number of orders per hour. Suppose that at $q = 40$, $T'(40) > 0$ and $T''(40) > 0$.
What can you say in words? A good answer is not “the derivative is positive.” It should say:

> *At 40 orders per hour, increasing demand will increase travel time, and the increase is itself accelerating. The system is moving toward congestion.*

Extreme Values and Critical Points

Core explanation, worked reasoning, and application

Concept: Extreme values

Let f be defined on a domain D.

- $f(c)$ is an **absolute maximum** if $f(c) \geq f(x)$ for all $x \in D$.
- $f(c)$ is an **absolute minimum** if $f(c) \leq f(x)$ for all $x \in D$.
- $f(c)$ is a **local maximum** or **local minimum** if the comparison only needs to hold near c.

A point c is a **critical point** if $f'(c) = 0$ or $f'(c)$ does not exist.

Concept: Closed-interval method

If f is continuous on $[a, b]$, then the **Extreme Value Theorem** guarantees an absolute maximum and an absolute minimum. To find them:

1. find all critical points in (a, b);
2. evaluate f at the critical points and endpoints;
3. compare the values.

Worked Example: Worked example 1: absolute extrema on a closed interval

Find the absolute maximum and minimum of

$$f(x) = 2x^3 - 3x^2 - 12x + 5 \qquad \text{on } [-2, 3].$$

First,

$$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1).$$

Critical points in $(-2, 3)$ are $x = -1$ and $x = 2$. Now compare the candidate values:

x	-2	-1	2	3
$f(x)$	1	12	-15	-4

Therefore the absolute maximum is 12 at $x = -1$, and the absolute minimum is -15 at $x = 2$.

Checkpoint: Checkpoint

Why do endpoints matter? Because an absolute maximum or minimum on a closed interval may occur at the boundary even when $f'(x) \neq 0$ there.

First Derivative Test: Direction of Change

Core explanation, worked reasoning, and application

Concept: Monotonicity from the sign of f'

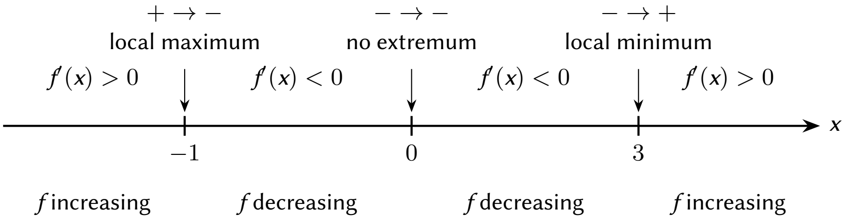
If $f'(x) > 0$ on an interval, then f is **increasing** there. If $f'(x) < 0$, then f is **decreasing**. This follows from the **Mean Value Theorem**.

Concept: First derivative test

Let c be a critical point where f is continuous.

- If f' changes from $+$ to $-$ at c, then $f(c)$ is a local maximum.
- If f' changes from $-$ to $+$ at c, then $f(c)$ is a local minimum.
- If f' does not change sign, then c is not a local extremum.

Illustration: First Derivative Test: Direction of Change



A first-derivative sign chart. Since $f'(x) > 0$ means f is increasing and $f'(x) < 0$ means f is decreasing, a change from $+$ to $-$ gives a local maximum, while a change from $-$ to $+$ gives a local minimum. At $x = 0$, the sign of f' does not change, so there is no extremum.

Worked Example: Worked example 2: first derivative test

Analyse local extrema of

$$f(x) = x^4 - 4x^3.$$

We compute

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3).$$

Critical points are $x = 0$ and $x = 3$. Since $4x^2 \geq 0$, the sign of $f'(x)$ away from 0 is determined by $x - 3$:

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
$f'(x)$	$-$	0	$-$	0	$+$
f	$\searrow$		$\searrow$		$\nearrow$

At $x = 0$, f' does not change sign, so there is no extremum. At $x = 3$, f' changes from negative to positive, so f has a local minimum, with $f(3) = -27$.

Concavity and the Second Derivative

Core explanation, worked reasoning, and application

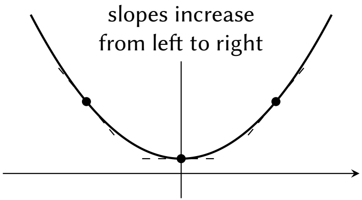
Concept: Concavity and inflection

The second derivative measures how the slope changes.

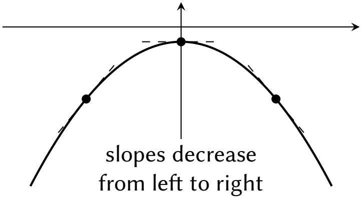
- If $f''(x) > 0$, then f' is increasing and the graph is **concave up**.
- If $f''(x) < 0$, then f' is decreasing and the graph is **concave down**.
- An **inflection point** occurs where concavity changes.

Illustration: Concavity and the Second Derivative

Concave up: $f''(x) > 0$



Concave down: $f''(x) < 0$



Concavity describes how tangent slopes change. For a concave-up graph, $f''(x) > 0$, the tangent slopes increase from left to right. For a concave-down graph, $f''(x) < 0$, the tangent slopes decrease from left to right.

Concept: Second derivative test

Suppose $f'(c) = 0$ and f'' is continuous near c.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, the test is inconclusive. Use the first derivative test.

Worked Example: Worked example 3: concavity and inflection

Find the inflection points of

$$f(x) = x^4 - 6x^2 + 1.$$

We have

$$f'(x) = 4x^3 - 12x, \qquad f''(x) = 12x^2 - 12 = 12(x-1)(x+1).$$

The possible inflection points are $x = -1$ and $x = 1$. Check the sign of f'':

x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f''(x)$	$+$	$-$	$+$
concavity	up	down	up

Concavity changes at both $x = -1$ and $x = 1$. Since $f(-1) = f(1) = -4$, the inflection points are $(-1, -4)$ and $(1, -4)$.

Curve Sketching as Model Diagnosis

Core explanation, worked reasoning, and application

Curve Sketching as Model Diagnosis

A curve sketch is not only a drawing exercise. In modeling, it diagnoses where a formula is valid, where it has unrealistic blow-up, where the response saturates, and where the model predicts a turning point.

Concept: Curve sketching checklist

For $y = f(x)$, analyse:

1. domain and excluded values;
2. intercepts and symmetry;
3. vertical, horizontal, or oblique asymptotes;
4. sign chart for f' and local extrema;
5. sign chart for f'' and inflection points;
6. interpretation in the original context.

Worked Example: Worked example 4: rational curve sketch

Sketch and interpret

$$f(x) = \frac{x^2}{x^2 - 1}.$$

The domain is $x \neq \pm 1$. The function is even. There are vertical asymptotes at $x = -1$ and $x = 1$, and

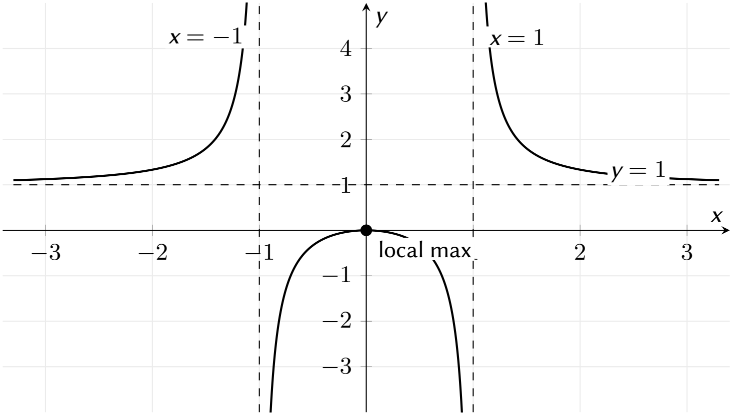
$$\lim_{x \to \pm\infty} \frac{x^2}{x^2 - 1} = 1,$$

so $y = 1$ is a horizontal asymptote. Also,

$$f'(x) = \frac{-2x}{(x^2 - 1)^2},$$

so f increases for $x < 0$ and decreases for $x > 0$ on each domain interval. The point $(0, 0)$ is a local maximum on the middle branch.

Worked Example: Worked example 4: rational curve sketch



A clean curve sketch for $f(x) = x^2/(x^2 - 1)$. Labels are placed outside dense regions and boxed with white background to avoid overlap with the branches and asymptotes.

Worked Example: Worked example 4: rational curve sketch

Model interpretation. If this formula represented a response ratio, then $x = \pm 1$ would be dangerous thresholds: the model predicts unbounded behaviour there. The horizontal asymptote $y = 1$ suggests the response stabilises far from the thresholds.

Model Critique: Model critique: derivative evidence

A derivative-based sketch is only as meaningful as the formula and domain. When using calculus in a modeling report, state:

- the domain used in the real problem;
- whether excluded points are physical boundaries or mathematical artefacts;
- whether maxima/minima occur inside the feasible range or at a boundary;
- whether a vertical asymptote indicates real danger, capacity failure, or a model breakdown.

Applied Optimization

Core explanation, worked reasoning, and application

Concept: Optimization workflow

A good optimization solution has four layers:

1. define variables with units;
2. write the objective and constraints;
3. reduce to a one-variable function on a realistic domain;
4. verify the optimum and interpret it in context.

Skipping the domain is one of the most common modeling mistakes.

Worked Example: Worked example 5: fencing beside a river

A farmer has 200 m of fencing and wants to enclose a rectangular field beside a straight river. No fence is needed along the river. Find the dimensions that maximize the area.
Let x be the side parallel to the river and y the width. Then

$$x + 2y = 200, \qquad A = xy.$$

Using $x = 200 - 2y$,

$$A(y) = (200 - 2y)y = 200y - 2y^2, \qquad 0 < y < 100.$$

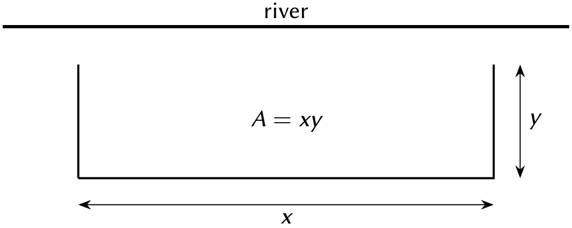
Now

$$A'(y) = 200 - 4y,$$

so $A'(y) = 0$ gives $y = 50$. Since $A''(y) = -4 < 0$, this gives a maximum. Therefore

$$y = 50 \text{ m}, \qquad x = 100 \text{ m}, \qquad A_{\max} = 5000 \text{ m}^2.$$

Worked Example: Worked example 5: fencing beside a river



$$\text{fencing used} = x + 2y$$

A rectangular field is built against a river, so only three sides need fencing. Thus $A = xy$ and the fencing constraint is $x + 2y = 200$.

Worked Example: Worked example 6: profit-maximizing production

A manufacturer sells q units per week. Cost is

$$C(q) = 0.01q^2 + 10q + 400,$$

and the price-demand equation is

$$p(q) = 50 - 0.02q.$$

Find the weekly production level maximizing profit.
Revenue is

$$R(q) = qp(q) = 50q - 0.02q^2.$$

Profit is

$$P(q) = R(q) - C(q) = (50q - 0.02q^2) - (0.01q^2 + 10q + 400) = 40q - 0.03q^2 - 400.$$

Then

$$P'(q) = 40 - 0.06q.$$

Set $P'(q) = 0$:

$$q = \frac{40}{0.06} \approx 666.7.$$

Since $P''(q) = -0.06 < 0$, this is a maximum for the continuous model. In practice, choose either $q = 666$ or $q = 667$ and compare integer profit.

Checkpoint: Checkpoint

Why might a continuous optimum be inappropriate for production quantity? Because real production may require an integer number of units, batch sizes, machine capacity, or demand uncertainty. The calculus result is a guide, not automatically the final decision.

Local Approximation and Newton's Method

Core explanation, worked reasoning, and application

Local Approximation and Newton's Method

For a differentiable function, the tangent line gives a local linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a).$$

This is useful when exact calculation is difficult, or when a model needs a quick sensitivity estimate.

Worked Example: Worked example 7: local linear approximation

Approximate $\sqrt{4.1}$ using $f(x) = \sqrt{x}$ near $a = 4$.

$$f(4) = 2, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}.$$

Therefore

$$\sqrt{x} \approx 2 + \frac{1}{4}(x - 4).$$

At $x = 4.1$,

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2.025.$$

This is close to the actual value $2.0248\ldots$.

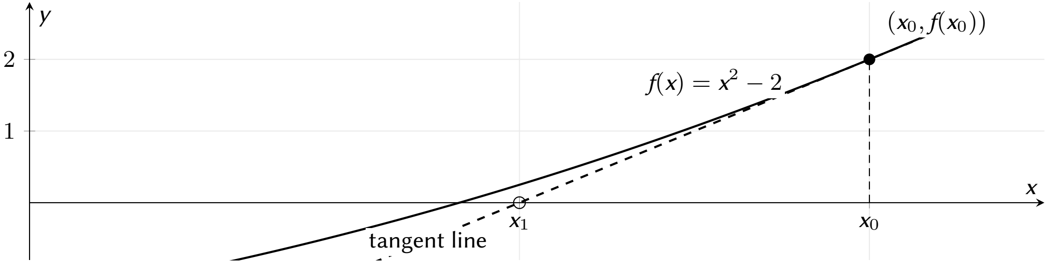
Concept: Newton's method

To solve $f(x) = 0$, start with a guess x_0 and iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Geometrically, x_{n+1} is the x-intercept of the tangent line at x_n.

Illustration: Local Approximation and Newton's Method



One step of Newton's method for $f(x) = x^2 - 2$. Starting from x_0, the tangent line meets the x-axis at the next approximation x_1.

Worked Example: Worked example 8: approximating $\sqrt{2}$

Solve $f(x) = x^2 - 2 = 0$ with $x_0 = 1$.

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

n	x_n
0	1.000000
1	1.500000
2	1.416667
3	1.414216
4	1.414214

Newton's method converges very quickly here because the initial guess is reasonable and the root is simple.

Differentiated Optimization Studio

Core explanation, worked reasoning, and application

Activity: 15-minute studio

Choose one route.
Core route. A box with square base and open top must have volume $32\ \mathrm{m}^3$. Find the dimensions minimizing surface area.
Challenge route. Find the largest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Extension route. Use Newton's method to approximate the positive solution of $\cos x = x$ starting from $x_0 = 0.7$. Explain why your iteration is credible.

Model Critique: How to write the modeling conclusion

A strong derivative-based conclusion should contain:

1. the objective being optimized;
2. the feasible domain;
3. the critical point or boundary comparison;
4. the verification method;
5. the practical interpretation and limitation.

For example:

> *Within the continuous production model, the profit function is concave because $P''(q) < 0$, so the critical point gives the global maximum. The optimal continuous production level is about 667 units per week. In an operational plan, this should be rounded only after checking integer profit and capacity constraints.*

Summary

Core explanation, worked reasoning, and application

Summary: Key takeaways

Derivative information	Modeling interpretation
$f'(x) > 0$	output increases as input increases.
$f'(x) < 0$	output decreases as input increases.
f' changes $+ \to -$	local maximum.
f' changes $- \to +$	local minimum.
$f''(x) > 0$	slope is increasing; response accelerates; concave up.
$f''(x) < 0$	slope is decreasing; diminishing returns or concave down.
Vertical asymptote	possible capacity limit, singularity, or model breakdown.
Closed interval comparison	absolute optimum may occur at a boundary.
Newton iteration	tangent-line method for solving equations numerically.

Exercises

Core explanation, worked reasoning, and application

Exercises: Core practice

1. Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on $[-1/2, 4]$.
2. For $f(x) = 2x^3 + 3x^2 - 36x + 5$, find critical points, intervals of increase/decrease, and local extrema.
3. Find all inflection points of $f(x) = x^5 - 5x^4 + 5x^3$.
4. Sketch $f(x) = \dfrac{x^2 - 4x + 3}{x - 2}$ using the curve-sketching checklist.
5. A manufacturer has cost $C(q) = 0.01q^2 + 10q + 400$ and price $p = 50 - 0.02q$. Rework the profit example and compare the integer choices $q = 666$ and $q = 667$.

Exercises: Modeling challenge

A cafeteria counter can serve at most 60 students per minute. A simple congestion model for average waiting time is

$$W(q) = \frac{q}{60 - q}, \qquad 0 < q < 60,$$

where q is arrival rate.

1. Find $W'(q)$ and $W''(q)$.
2. Interpret the signs of W' and W''.
3. What happens as $q \to 60^-$?
4. Write a short recommendation explaining why operating near capacity is risky.

Exercises: Extension practice

1. Use Newton's method to find the positive root of $x^3 + 2x - 5 = 0$ starting from $x_0 = 1$.
2. Find the Maclaurin polynomial $P_4(x)$ for $\cos x$ and use it to approximate $\cos(0.2)$.
3. Find the dimensions of the largest rectangle inscribed under the semicircle $x^2 + y^2 = R^2$, $y \geq 0$.