Supplementary 5: Derivatives, Rates, and Local Linear Models

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this supplementary lesson, readers should be able to:

1. interpret the **derivative** as a local slope, instantaneous rate, and sensitivity coefficient;
2. compute simple derivatives from the **difference quotient**;
3. use the power, product, quotient, trigonometric, exponential, logarithmic, and **chain rule** correctly;
4. identify common reasons why a function is not differentiable;
5. use implicit, parametric, and related-rate differentiation in geometric or physical models;
6. use the **Mean Value Theorem** to connect average and instantaneous rates;
7. explain derivative calculations in a modeling report instead of presenting them as isolated algebra.

Derivative as slope, rate, and sensitivity

Core explanation, worked reasoning, and application

Derivative as slope, rate, and sensitivity

In modeling, a derivative is not just a calculus operation. It answers a practical question: *how fast does the output change when the input changes slightly?* If $y = f(x)$, then $f'(a)$ measures the local change in y per unit change in x near $x = a$.

Activity: Launch: average speed versus speed at one instant

A robot travels along a corridor. Its distance from the start after t seconds is modeled by

$$s(t) = 0.4t^2 + 1.2t \qquad (0 \le t \le 10),$$

where s is in metres.

1. Compute the average speed from $t = 3$ to $t = 5$.
2. Compute the average speed from $t = 4$ to $t = 4.1$.
3. What value should the speedometer display at exactly $t = 4$?

Discussion point: smaller time windows give more local information. The derivative is the limiting value of these average rates.

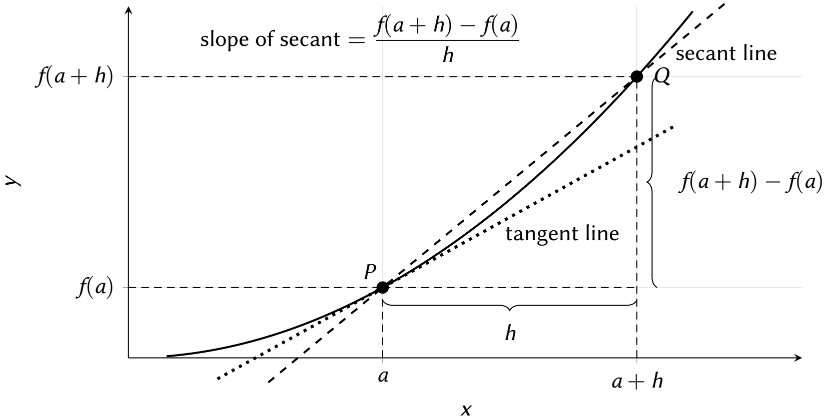
Concept: Core definition

For a function f, the **derivative at** a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. The fraction is the **difference quotient**. It is the slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$.

Illustration: Derivative as slope, rate, and sensitivity



The derivative at a is the limit of the slopes of secant lines: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. As h becomes smaller, the secant line through P and Q approaches the tangent line at P.

Worked Example: Worked example 1: speed from a position model

For $s(t) = 0.4t^2 + 1.2t$, the instantaneous speed at $t = 4$ is

$$s'(4) = \lim_{h \to 0} \frac{s(4+h) - s(4)}{h}.$$

Compute

$$\begin{aligned} s(4+h) &= 0.4(4+h)^2 + 1.2(4+h) \\ &= 0.4(16 + 8h + h^2) + 4.8 + 1.2h \\ &= 11.2 + 4.4h + 0.4h^2, \end{aligned}$$

while $s(4) = 11.2$. Hence

$$\frac{s(4+h) - s(4)}{h} = 4.4 + 0.4h \to 4.4.$$

So the robot's speed at $t = 4$ is 4.4 m/s.

Checkpoint: Checkpoint 1

Explain the units of $s'(4)$. Why is it not measured in metres?

First-principles derivatives

Core explanation, worked reasoning, and application

First-principles derivatives

Before using rules, students should experience why the rules are true. This prevents derivatives from becoming a memorised table detached from modeling.

Worked Example: Worked example 2: derivative of x^2 from the definition

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

At $x = 3$, the local slope is 6. This means that near $x = 3$, increasing x by 0.01 increases x^2 by about 0.06.

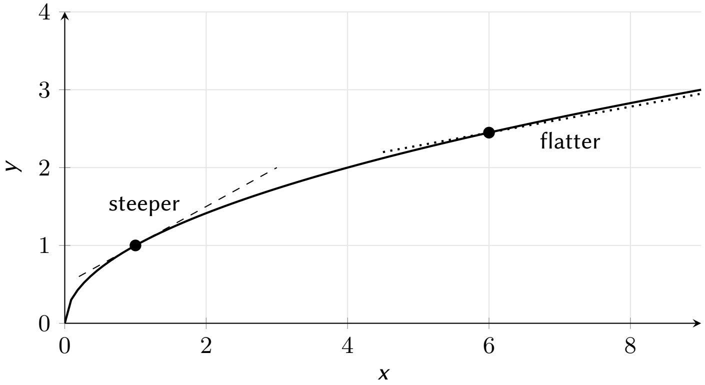
Worked Example: Worked example 3: derivative of $\sqrt{x}$

For $x > 0$,

$$\begin{aligned} \frac{d}{dx}\sqrt{x} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

The derivative is larger near $x = 0$, so the square-root graph is steep near the origin and gradually flattens.

Illustration: First-principles derivatives



The derivative of $\sqrt{x}$ decreases as x increases. This is a useful visual interpretation of $1/(2\sqrt{x})$.

Model Critique: Model critique: derivative as local linear model

A derivative gives a local approximation:

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x.$$

This is powerful, but it is local. It may be poor when Δx is large or when the graph bends sharply. In a modeling report, always state the interval over which the local approximation is being used.

Differentiation rules for efficient calculation

Core explanation, worked reasoning, and application

Concept: Core derivative rules

If f and g are differentiable, then

$$\begin{aligned}
&\frac{d}{dx}(c) = 0, && \frac{d}{dx}(x^n) = nx^{n-1}, && \frac{d}{dx}[cf] = cf', \\
&\frac{d}{dx}(f+g) = f' + g', && \frac{d}{dx}(fg) = f'g + fg', && \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.
\end{aligned}$$

The product rule is not $(fg)' = f'g'$. The quotient rule is not the derivative of numerator divided by derivative of denominator.

Worked Example: Worked example 4: product and quotient rules

Differentiate

$$F(x) = x^2 \sin x + \frac{x^2+1}{x-1}.$$

For the product term,

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x.$$

For the quotient term,

$$\frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{2x(x-1)-(x^2+1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}.$$

Therefore

$$F'(x) = 2x \sin x + x^2 \cos x + \frac{x^2-2x-1}{(x-1)^2}.$$

Worked Example: Worked example 5: trigonometric and exponential derivatives

For a seasonal demand model

$$D(t) = 120 + 30\sin\left(\frac{2\pi t}{12}\right) + 5e^{0.03t},$$

where t is measured in months, compute the instantaneous rate of change:

$$D'(t) = 30\cos\left(\frac{2\pi t}{12}\right)\cdot\frac{2\pi}{12} + 5e^{0.03t}\cdot 0.03.$$

Thus

$$D'(t) = 5\pi\cos\left(\frac{\pi t}{6}\right) + 0.15e^{0.03t}.$$

The first term describes seasonal increase/decrease; the second term describes long-term growth.

Checkpoint: Checkpoint 2

Why is the derivative of $\sin(\pi t/6)$ not simply $\cos(\pi t/6)$? Name the rule being used.

The chain rule as a modeling pipeline

Core explanation, worked reasoning, and application

The chain rule as a modeling pipeline

The **chain rule** is essential for modeling because many models are pipelines: an input affects an intermediate quantity, which affects the output.

Concept: Chain rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In function notation,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Illustration: The chain rule as a modeling pipeline

$$\boxed{\begin{array}{c}\text{input}\\ x\end{array}} \xrightarrow{\frac{du}{dx} = g'(x)} \boxed{\begin{array}{c}\text{inner function}\\ u = g(x)\end{array}} \xrightarrow{\frac{dy}{du} = f'(u)} \boxed{\begin{array}{c}\text{outer function}\\ y = f(u) = f(g(x))\end{array}}$$

$$\boxed{\begin{array}{c}\text{Overall rate from } x \text{ to } y\\ \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = f'(g(x))\,g'(x)\end{array}}$$

The chain rule for a composite function $y = f(g(x))$. The input x first changes the intermediate variable $u = g(x)$, and then u changes the final output $y = f(u)$. Thus the overall rate is the product of the two rates.

Worked Example: Worked example 6: pollution concentration through a transformation

A concentration index is modeled by

$$I(t) = \sqrt{1 + 0.4C(t)}, \qquad C(t) = 20e^{-0.08t}.$$

Find $I'(t)$ and interpret the sign.

$$I'(t) = \frac{1}{2\sqrt{1 + 0.4C(t)}} \cdot 0.4C'(t).$$

Since $C'(t) = -1.6e^{-0.08t}$,

$$I'(t) = \frac{0.2(-1.6e^{-0.08t})}{\sqrt{1 + 8e^{-0.08t}}} < 0.$$

The index is decreasing. The derivative becomes closer to zero over time because the exponential decay slows in absolute size.

Worked Example: Worked example 7: logarithmic differentiation

Differentiate $y = x^x$ for $x > 0$.

$$\ln y = x \ln x.$$

Differentiate both sides:

$$\frac{y'}{y} = \ln x + 1.$$

Therefore

$$y' = x^x(\ln x + 1).$$

This method is useful when both the base and exponent vary.

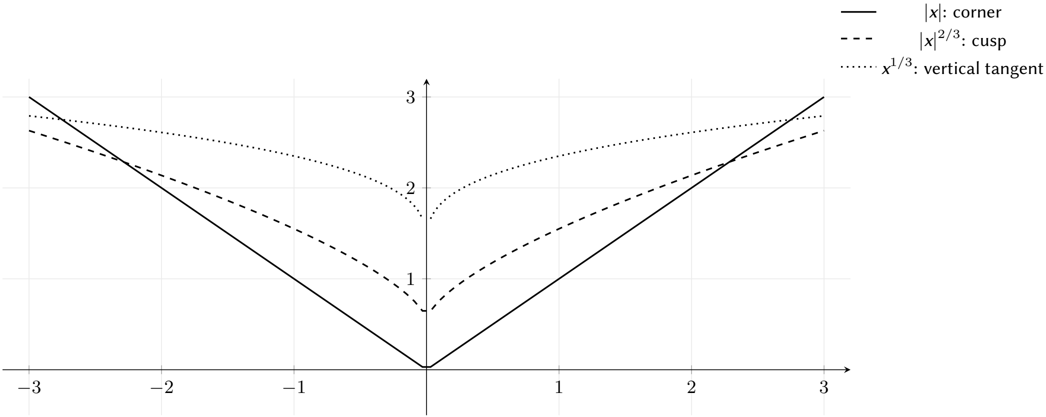
When derivatives do not exist

Core explanation, worked reasoning, and application

When derivatives do not exist

A derivative may fail to exist even when a graph looks simple. This matters in modeling: a non-differentiable point may represent a switch, threshold, collision, capacity boundary, or change of policy.

Illustration: When derivatives do not exist



Three common non-differentiable behaviours. Each can have a modeling interpretation.

Checkpoint: Checkpoint 3

For $f(x) = |x|$, compute the right-hand derivative and left-hand derivative at 0. Why does the derivative not exist there?

Model Critique: Model critique: smoothing versus preserving a threshold

In applications, replacing a kink by a smooth curve may make calculus easier, but it may erase a real threshold. For example, a penalty fee may start exactly when demand exceeds capacity. A good modeler should ask: is the corner a mathematical inconvenience, or is it the phenomenon we need to preserve?

Implicit, parametric, and related-rate differentiation

Core explanation, worked reasoning, and application

Implicit, parametric, and related-rate differentiation

Not all models are naturally written as $y = f(x)$. Circles, ellipses, supply-demand equilibria, and geometric constraints often appear as equations involving several variables.

Worked Example: Worked example 8: implicit differentiation on a circle

For a circle

$$x^2 + y^2 = 25,$$

we differentiate both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 0,$$

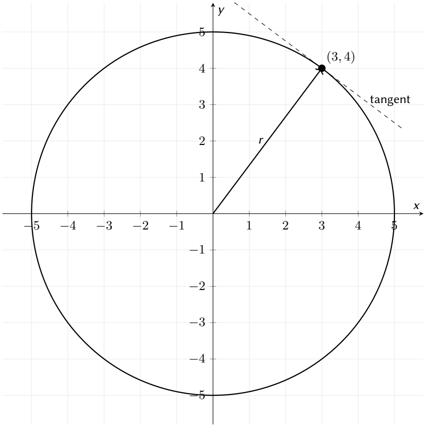
so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

At $(3, 4)$, the tangent slope is $-3/4$, so the tangent line is

$$y - 4 = -\frac{3}{4}(x - 3).$$

Illustration: Implicit, parametric, and related-rate differentiation



Implicit differentiation finds tangent slopes even when the curve is not a function of x globally.

Concept: Parametric derivative

If a curve is given by $x = g(t)$ and $y = h(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad dx/dt \neq 0.$$

This is useful for motion models because t is the natural independent variable.

Worked Example: Worked example 9: related rates for a sliding ladder

A 10 m ladder leans against a wall. Its base moves away at 0.5 m/s. How fast is the top moving down when the base is 6 m from the wall?

Let x be the distance of the base from the wall and y the height of the top. Then

$$x^2 + y^2 = 100.$$

Differentiate with respect to time:

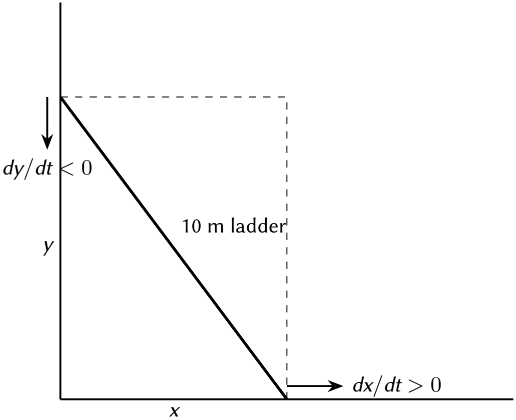
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

When $x = 6$, $y = 8$. Hence

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8}(0.5) = -0.375 \text{ m/s}.$$

The negative sign means the height is decreasing.

Illustration: Implicit, parametric, and related-rate differentiation



Related rates translate a geometric constraint into a relationship between rates.

Mean Value Theorem and local linear thinking

Core explanation, worked reasoning, and application

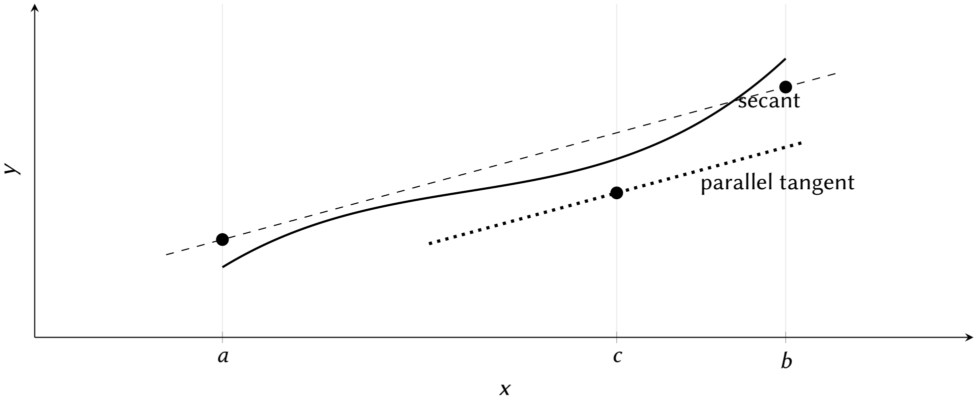
Concept: Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The instantaneous rate somewhere equals the average rate over the interval.

Illustration: Mean Value Theorem and local linear thinking



The Mean Value Theorem links an average rate to at least one instantaneous rate.

Worked Example: Worked example 10: interpreting an average speed claim

A car travels 120 km in 1.5 hours. Assuming the position function is continuous and differentiable, the Mean Value Theorem says that at some instant the car's speed was exactly

$$\frac{120}{1.5} = 80 \text{ km/h}.$$

This does not say the speed was always 80 km/h. It says an instantaneous value equal to the average must have occurred at least once.

Summary: What students should take away

- A derivative is a local rate, slope, and sensitivity.
- The difference quotient is the bridge from average rate to instantaneous rate.
- Rules speed up calculation, but the chain rule is the main rule for composite models.
- Non-differentiability is often meaningful, not merely a technical failure.
- Implicit and related-rate methods are essential when the model is constrained geometrically.
- The MVT connects local and average behaviour, which is why derivatives support real-world interpretation.

Optional extension: L'Hôpital's Rule

Core explanation, worked reasoning, and application

Concept: Reserve material

L'Hôpital's Rule belongs after students understand derivatives. If a limit has indeterminate form $0/0$ or ∞/∞, and the relevant derivative quotient limit exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

It should not be used before checking the indeterminate form.

Worked Example: Optional example: exponential dominates polynomial

Evaluate

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

The form is ∞/∞. Applying L'Hôpital twice gives

$$\lim_{x\to\infty} \frac{2x}{e^x} o \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Thus exponential growth eventually dominates quadratic growth.

Exercises

Core explanation, worked reasoning, and application

Exercises: Core practice

1. Use the definition of derivative to find the derivative of $f(x) = 3x^2 - 2x$.
2. Differentiate:
$$f(x) = 5x^4 - 3x^{-2} + 7\sqrt{x} - 9.$$
3. Differentiate $g(x) = (x^2 + 1)(\sin x + x)$.
4. Differentiate $h(x) = \dfrac{x^2 + 3x}{x - 2}$.
5. Differentiate $p(x) = \sin(3x^2 + 1)$ and explain which part is the inner function.
6. Find the tangent line to $x^2 + y^2 = 13$ at $(2, 3)$.

Exercises: Modeling practice

1. A queue waiting-time model is $W(\rho) = \dfrac{4\rho}{1-\rho}$ for $0 < \rho < 1$. Compute $W'(\rho)$ and explain what happens as $\rho \to 1^-$. Why is this important for capacity planning?
2. A pollutant concentration follows $C(t) = 60e^{-0.12t} + 8$. Compute $C'(t)$ and determine when the magnitude of the rate first falls below 1 unit/day.
3. A circular infection front has radius $r(t)$ and area $A(t) = \pi r(t)^2$. If $dr/dt = 0.3$ km/day, find dA/dt when $r = 5$ km.
4. A seasonal sales model is $S(t) = 200 + 40\cos(\pi t/6)$. Find when sales are increasing fastest and interpret your answer.

Exercises: Challenge and extension

1. Show that if $f'(x) = 0$ for all x in an interval, then f is constant on that interval. State clearly where the Mean Value Theorem is used.
2. Differentiate $y = x^x \sin x$ for $x > 0$.
3. For the parametric curve $x = t^2 + 1$, $y = t^3 - t$, find dy/dx and the slope at $t = 2$.
4. Use L'Hôpital's Rule to evaluate
$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$
5. Write a short modeling paragraph explaining why a derivative-based approximation may fail near a kink or threshold.