Supplementary 4: Limits, Continuity, and Asymptotic Thinking
Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this supplementary lesson, readers should be able to:

1. explain a **limit** as the value approached by a function near a point, not necessarily at the point;
2. evaluate common limits by substitution, factorisation, rationalisation, and standard trigonometric limits;
3. use **one-sided limits** to decide whether a two-sided limit exists;
4. interpret **infinite limits**, **limits at infinity**, and **asymptotes**;
5. classify removable, jump, and infinite discontinuities;
6. apply **continuity**, the **Intermediate Value Theorem**, and the **Extreme Value Theorem** in modeling contexts;
7. explain numerical and graphical evidence for a limit without confusing it with proof.

Limits as local behaviour

Core explanation, worked reasoning, and application

Limits as local behaviour

A **limit** describes what a function is approaching. It is local: it looks near a point, not necessarily at the point. This idea is central to calculus, but it is also a modeling idea. A sensor may fail at one time, a formula may be undefined at one value, or a process may approach a capacity without ever reaching it exactly.

Activity: Launch: the missing sensor reading

A temperature sensor records readings every minute. At $t = 5$, the sensor fails, but nearby readings suggest the curve should pass near 23.8°C.

1. Should we say the temperature at $t = 5$ is known?
2. Should we say the *trend* near $t = 5$ is known?
3. What evidence would make the estimated limiting value more credible?

Modeling lesson: a limit is about approaching behaviour. It can be meaningful even when the value at the exact input is missing or unreliable.

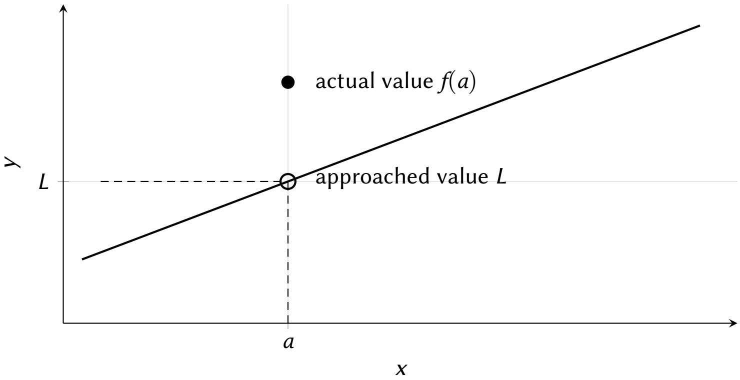
Concept: Core definition, stated informally

We write

$$\lim_{x \to a} f(x) = L$$

when $f(x)$ can be made as close as we like to L by choosing x sufficiently close to a, with $x \neq a$. The value $f(a)$ may equal L, differ from L, or fail to exist.

Illustration: Limits as local behaviour



A limit describes the value approached by the graph near $x = a$. The actual value at a can be different.

Worked Example: Worked Example 1: A removable hole

Evaluate

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

At $x = 1$, the expression is undefined. But for $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1.$$

Therefore

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

Interpretation: the original formula has a hole at $x = 1$, but the nearby trend approaches 2.

Worked Example: Worked Example 2: Rationalisation

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}.$$

Direct substitution gives $0/0$, so we rationalise:

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{x}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2}, \qquad x \neq 0.$$

Hence

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{1}{2+2} = \frac{1}{4}.$$

Checkpoint: Algebraic limit check

Evaluate:

1. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$;
2. $\lim_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$;
3. $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$.

For each one, state the obstacle first: direct substitution, $0/0$, square root, or factorisation.

From informal limits to epsilon-delta language

Core explanation, worked reasoning, and application

From informal limits to epsilon-delta language

In a first modeling course, students do not need to master every ε-δ **proof**. They should, however, understand the meaning of the definition: output tolerance controls input tolerance.

Concept: The formal definition

We say

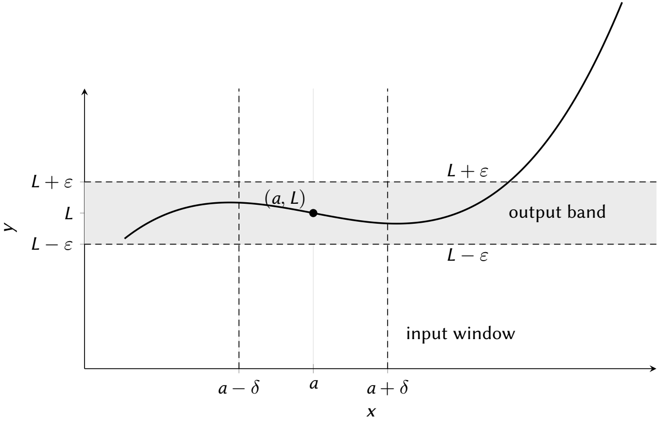
$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Here ε is the allowed error in the output, while δ is the allowed distance from the input a.

Illustration: From informal limits to epsilon-delta language



The epsilon-delta definition of a limit: if x is sufficiently close to a, then $f(x)$ is within ε of L.

Worked Example: Worked Example 3: A simple epsilon-delta proof

Prove that $\lim_{x\to 3}(2x-1)=5$.
We want

$$|(2x-1)-5| = |2x-6| = 2\,|x-3| < \varepsilon.$$

So it is enough to require $|x-3|<\varepsilon/2$. Let $\delta=\varepsilon/2$. Then whenever $0<|x-3|<\delta$,

$$|(2x-1)-5| = 2\,|x-3| < 2\delta = \varepsilon.$$

Therefore $\lim_{x\to 3}(2x-1)=5$.

Model Critique: Proof versus evidence

A numerical table can suggest a limit; a graph can illustrate a limit; an epsilon-delta proof establishes the limit.
In mathematical modeling reports, students usually need to explain the trend and justify the computation.
Formal proof is useful when the result will be used as a theorem or when the graph is misleading.

One-sided and infinite limits

Core explanation, worked reasoning, and application

One-sided and infinite limits

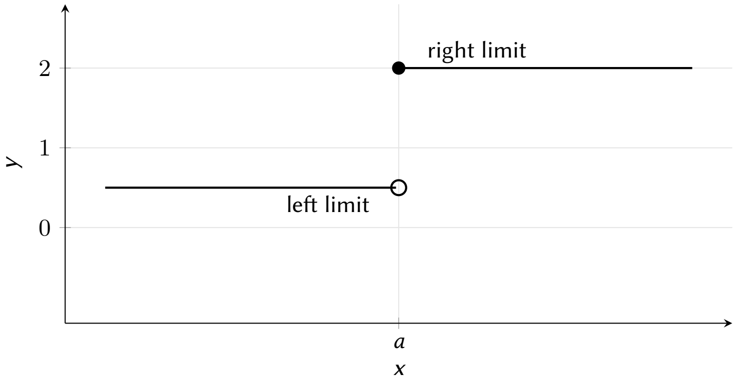
Some functions approach different values from the left and from the right. A two-sided limit exists only when the two one-sided limits agree.

Concept: One-sided limits

The **right-hand limit** is $\lim_{x \to a^+} f(x)$; it approaches a using $x > a$. The **left-hand limit** is $\lim_{x \to a^-} f(x)$; it approaches a using $x < a$.

$$\lim_{x \to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Illustration: One-sided and infinite limits



A jump occurs when the left-hand and right-hand limits are different.

Worked Example: Worked Example 4: Absolute value quotient

Evaluate

$$\lim_{x \to 0} \frac{|x|}{x}.$$

If $x > 0$, then $|x| = x$, so $|x|/x = 1$. If $x < 0$, then $|x| = -x$, so $|x|/x = -1$. Therefore

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since the two one-sided limits are unequal, the two-sided limit does not exist.

Concept: Infinite limits and vertical asymptotes

Writing $\lim_{x \to a} f(x) = +\infty$ means that $f(x)$ grows without bound as x approaches a. It does not mean that $+\infty$ is an ordinary real-number limit. If at least one one-sided limit is infinite, then the line $x = a$ is a **vertical asymptote**.

Worked Example: Worked Example 5: Capacity blow-up

A simple congestion model for waiting time is

$$W(q) = \frac{1}{1-q}, \qquad 0 \le q < 1,$$

where q is demand as a fraction of capacity. Then

$$\lim_{q \to 1^-} W(q) = +\infty.$$

Interpretation: as demand approaches full capacity, the model predicts waiting time grows extremely fast. The vertical asymptote $q = 1$ is a warning that operating close to capacity is unstable.

Limits at infinity and asymptotic thinking

Core explanation, worked reasoning, and application

Limits at infinity and asymptotic thinking

A **limit at infinity** describes long-run behaviour. In modeling, it answers questions such as: Does the prediction stabilise? Does a cost approach a baseline? Does a probability approach one?

Worked Example: Worked Example 6: Long-run behaviour of a rational model

Evaluate

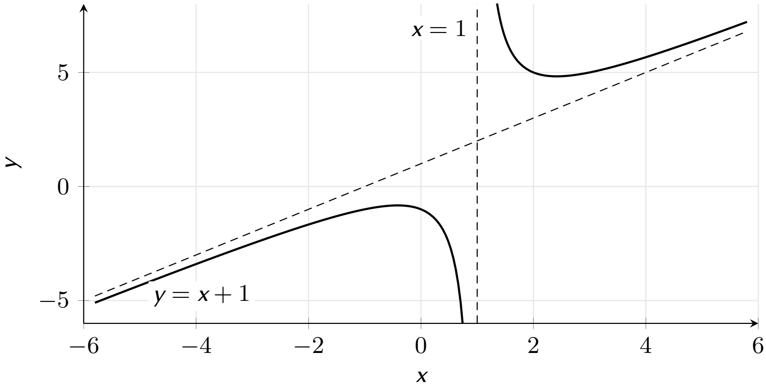
$$\lim_{x\to\infty} \frac{3x^2 - 5x + 2}{2x^2 + x - 1}.$$

Divide numerator and denominator by x^2:

$$\frac{3 - 5/x + 2/x^2}{2 + 1/x - 1/x^2} \to \frac{3}{2}.$$

Therefore the model has horizontal asymptote $y = 3/2$.

Illustration: Limits at infinity and asymptotic thinking



Asymptotes summarise behaviour near forbidden inputs and far from the origin.

Summary: Limit techniques for class use

Situation	**Technique**	**Example**
Continuous expression	Substitute directly	polynomial, square root on domain
$0/0$ with polynomial	Factor and cancel	$(x^2-4)/(x-2)$
$0/0$ with radicals	Rationalise	$(\sqrt{x+4}-2)/x$
Oscillating bounded factor	Squeeze theorem	$x^2\sin(1/x)$
Piecewise or absolute value	Check one-sided limits	$\lvert x\rvert/x$
Rational function at infinity	Divide by highest power	leading coefficient ratio
Vertical blow-up	One-sided infinite limits	$1/(x-a)$ or $1/(x-a)^2$

Continuity: no jump, no hole, no blow-up

Core explanation, worked reasoning, and application

Continuity: no jump, no hole, no blow-up

Continuity means that the function value matches the limiting trend.

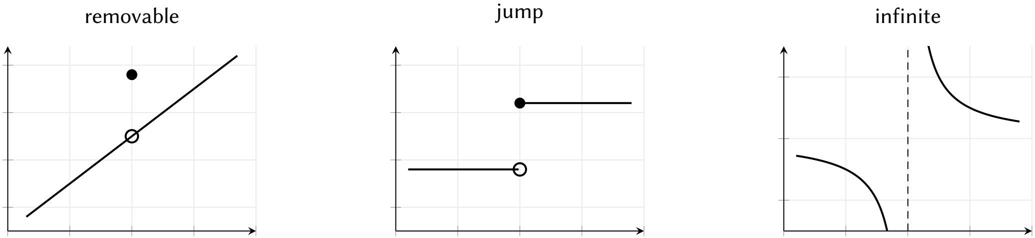
Concept: Continuity at a point

A function f is **continuous** at $x = a$ if all three conditions hold:

1. $f(a)$ is defined;
2. $\lim_{x\to a} f(x)$ exists;
3. $\lim_{x\to a} f(x) = f(a)$.

If any condition fails, the function is discontinuous at a.

Illustration: Continuity: no jump, no hole, no blow-up



Three common discontinuities: a hole, a jump, and a vertical blow-up.

Worked Example: Worked Example 7: Make a piecewise model continuous

Find k so that

$$f(x) = \begin{cases} kx^2 - 1, & x < 2, \\ 3x + k, & x \geq 2 \end{cases}$$

is continuous at $x = 2$.
Left-hand limit:

$$\lim_{x \to 2^-} f(x) = 4k - 1.$$

Right-hand value and limit:

$$\lim_{x \to 2^+} f(x) = f(2) = 6 + k.$$

Set them equal:

$$4k - 1 = 6 + k \quad \Rightarrow \quad 3k = 7 \quad \Rightarrow \quad k = \frac{7}{3}.$$

Model Critique: Modeling interpretation of continuity

Continuity is an assumption. A physical temperature curve is often continuous; a fare system, school grade boundary, or traffic-light rule may have jumps. Before applying a continuous model, ask whether the real system changes smoothly or by thresholds.

Existence theorems: IVT and EVT

Core explanation, worked reasoning, and application

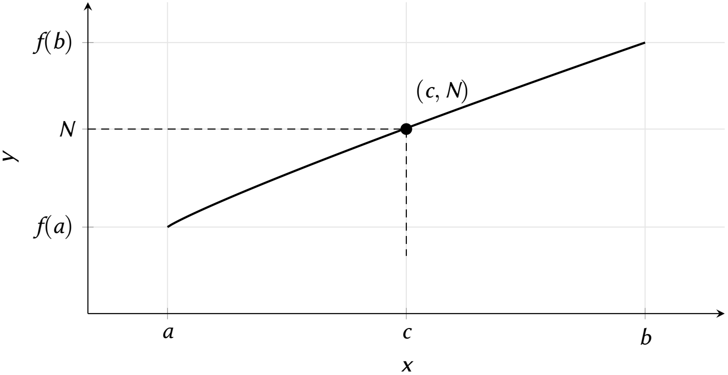
Existence theorems: IVT and EVT

The **Intermediate Value Theorem** and **Extreme Value Theorem** are important because they guarantee existence even when we cannot solve exactly.

Concept: Intermediate Value Theorem

If f is continuous on $[a, b]$ and N lies between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = N$.
In particular, if $f(a)$ and $f(b)$ have opposite signs, then f has a root in (a, b).

Illustration: Existence theorems: IVT and EVT



A continuous graph cannot move from $f(a)$ to $f(b)$ without taking every intermediate value.

Worked Example: Worked Example 8: Existence of a solution

Show that

$$x^3 - x - 1 = 0$$

has a solution between 1 and 2.
Let $f(x) = x^3 - x - 1$. Since f is a polynomial, it is continuous. Compute

$$f(1) = 1 - 1 - 1 = -1 < 0, \qquad f(2) = 8 - 2 - 1 = 5 > 0.$$

By the IVT, there exists $c \in (1, 2)$ such that $f(c) = 0$.

Concept: Extreme Value Theorem

If f is continuous on a closed bounded interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$.

For modeling, this theorem justifies the phrase “there is an optimal value” before we try to find it. The hypotheses matter: continuous function, closed interval, bounded interval.

Squeeze theorem and trigonometric limits

Core explanation, worked reasoning, and application

Squeeze theorem and trigonometric limits

The **Squeeze Theorem** handles functions trapped between two simpler functions.

Concept: Squeeze theorem

If

$$g(x) \leq f(x) \leq h(x)$$

near a, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} f(x) = L$.

Worked Example: Worked Example 9: Oscillation controlled by a vanishing factor

Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Since $-1 \leq \sin(1/x) \leq 1$ for $x \neq 0$,

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Both outer functions approach 0, so by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Summary: Two standard trigonometric limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

These are foundational for derivatives of trigonometric functions. They also show why radians, not degrees, are the natural angle unit in calculus.

Worked Example: Worked Example 10: Using the standard trig limit

Evaluate

$$\lim_{x \to 0} \frac{\sin(3x)}{x}.$$

Rewrite:

$$\frac{\sin(3x)}{x} = 3 \cdot \frac{\sin(3x)}{3x}.$$

As $x \to 0$, also $3x \to 0$, so

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3 \cdot 1 = 3.$$

Differentiated limit studio

Core explanation, worked reasoning, and application

Activity: Differentiated studio

Choose one route.

1. **Core route: algebraic limits.** Evaluate

$$\lim_{x\to -2}\frac{x^3+8}{x+2},\qquad \lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}.$$

2. **Challenge route: one-sided and continuity.** For

$$f(x)=\begin{cases} ax+1, & x<1,\\ x^2+a, & x\geq 1,\end{cases}$$

find a so that f is continuous at $x=1$.

3. **Extension route: asymptotic model.** A cost model is

$$C(n)=\frac{5n^2+30n}{2n^2+1}.$$

Find the long-run cost and explain what the horizontal asymptote means.

Model Critique: Competition-style communication

When using limits in a modeling report, write in context.

Weak statement: "The limit is infinity."

Stronger statement: "As demand approaches full capacity from below, the waiting-time model $W(q) = 1/(1-q)$ grows without bound. This suggests that operating near $q = 1$ is unsafe: small increases in demand may produce very large waiting times. The conclusion depends on the model's assumption that service capacity remains fixed."

Summary: Lesson summary

Concept	Modeling meaning
Limit	Describes approaching behaviour near a point.
Hole	Formula may be undefined at one input while the trend remains clear.
One-sided limit	Detects thresholds, jumps, and directional behaviour.
Infinite limit	Signals blow-up or capacity failure.
Limit at infinity	Describes long-run behaviour and horizontal asymptotes.
Continuity	Assumes smooth change; may fail in threshold systems.
IVT	Guarantees existence of a solution between two signs or values.
EVT	Guarantees a maximum/minimum when continuity and closed interval hold.
Squeeze theorem	Controls a complicated function between simpler ones.

Checkpoint: Exit ticket

In three sentences, explain the difference between $f(a)$ and $\lim_{x \to a} f(x)$. Then give one modeling situation where the limit is useful even if $f(a)$ is missing.

Exercises: Core practice

1. Evaluate $\lim_{x \to 4} (5x - 3)$ using direct substitution.
2. Evaluate $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$.
3. Evaluate $\lim_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$.
4. Evaluate $\lim_{x \to 0} \dfrac{\tan x}{x}$.
5. Find the one-sided limits of $f(x) = \lfloor x \rfloor$ at $x = 3$.
6. Classify the discontinuity of $f(x) = (x^2 - 9)/(x - 3)$ at $x = 3$.

Exercises: Modeling challenge

A school cafeteria uses

$$W(q) = \frac{4q}{1-q}, \qquad 0 \le q < 1,$$

to estimate average waiting time in minutes, where q is demand divided by service capacity.

1. Find $\lim_{q\to 0^+} W(q)$ and interpret.
2. Find $\lim_{q\to 1^-} W(q)$ and interpret.
3. Find the value of q at which the model predicts $W = 12$ minutes.
4. State one limitation of this model.
5. Write a short recommendation for cafeteria managers based on the limiting behaviour.

Exercises: Extension practice

1. Prove using the epsilon-delta definition that $\lim_{x\to 1}(3x+2)=5$.
2. Use the sequential criterion to show that $\lim_{x\to 0}\sin(1/x)$ does not exist.
3. Evaluate $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x+3}-x\right)$.
4. Find all vertical and horizontal asymptotes of $f(x)=\dfrac{3x^2-2}{x^2-4}$.
5. Use the IVT to prove that $e^x=3-2x$ has at least one real solution.

Summary: Optional appendix: L'Hopital's Rule preview

Some limits produce indeterminate forms such as $0/0$ or ∞/∞. After derivatives are introduced, one powerful tool is L'Hopital's Rule: under suitable conditions,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

This rule should not replace algebraic thinking. Factorisation, rationalisation, and standard trigonometric limits remain important because they reveal the structure of the expression.