Supplementary 2: Numbers, Coordinates, and Geometric Models

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this supplementary lesson, readers should be able to:

1. distinguish **rational numbers**, **irrational numbers**, and **real numbers**;
2. interpret **absolute value** as distance and use it to describe tolerances or errors;
3. represent intervals on the **real number line**;
4. use the coordinate plane to model location, distance, midpoint, and coverage;
5. derive and interpret equations of lines and circles;
6. connect slope, distance, and geometry to data fitting, routing, optimization, and modeling constraints.

Why numbers and coordinates matter in modeling

Core explanation, worked reasoning, and application

Why numbers and coordinates matter in modeling

A mathematical model turns a real situation into objects that can be computed, compared, and communicated. Before calculus, optimization, or simulation can be used, students need a precise language for three simple ideas:

$$\text{quantity}, \qquad \text{location}, \qquad \text{distance}.$$

Numbers describe quantities, coordinates describe location, and distance formulas describe closeness, error, travel, or coverage. This supplementary note is therefore not a separate algebra review; it is the coordinate language behind many modeling tasks.

Activity: Launch: choosing the right number system

For each modeling quantity, decide whether integers, rational numbers, or real numbers are the most natural language. Give one sentence of explanation.

1. Number of students assigned to a team.
2. Proportion of students who chose bus travel.
3. Distance between two locations on a map.
4. Error between an observed value and a predicted value.
5. Number of delivery robots required for a school route.

Modeling lesson: Choosing the wrong number system can create wrong decisions. For example, a linear program may output 2.7 robots, but the actual decision must be an integer.

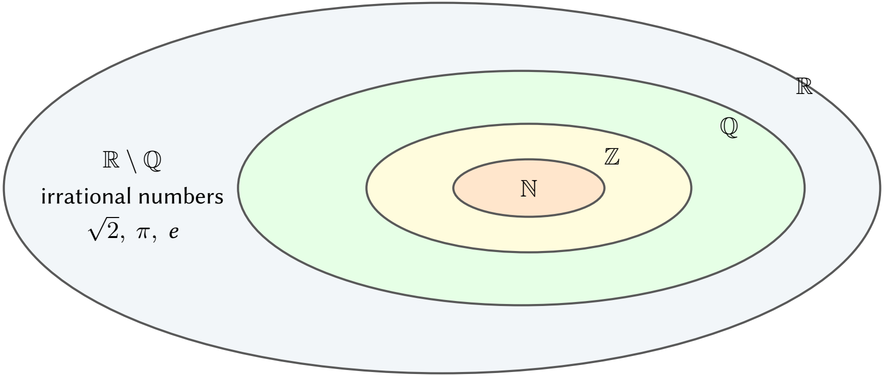
Number systems: from counting to the real line

Core explanation, worked reasoning, and application

Concept: Core vocabulary

Symbol	Name	Modeling meaning
$\mathbb{N}$	natural numbers	counting objects: $1, 2, 3, \ldots$
$\mathbb{Z}$	integers	signed counts or net changes: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	rational numbers	ratios, fractions, exact proportions: m/n with $n \neq 0$
$\mathbb{R}$	real numbers	continuous quantities: length, time, mass, temperature, error

Illustration: Number systems: from counting to the real line



The hierarchy of number systems. The natural numbers are contained in the integers, the integers are contained in the rational numbers, and the rational numbers are contained in the real numbers. The irrational numbers are the real numbers that are not rational.

Worked Example: Worked Example 1: Classifying modeling quantities

Classify each quantity.

1. A school needs 7 buses. This is in $\mathbb{N}$ because it counts objects.
2. A team completed $\frac{3}{4}$ of its survey. This is in $\mathbb{Q}$ because it is a ratio of integers.
3. The diagonal of a 1×1 square is $\sqrt{2}$. This is irrational, hence real but not rational.
4. A fitted parameter $k = 0.03716\ldots$ from regression is usually treated as real.

Modeling interpretation: Some models are continuous approximations of discrete decisions. That is allowed, but the report must explain how the continuous result is converted into a feasible decision.

Concept: Rational numbers and the gap problem

A **rational number** is a number of the form m/n, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Rational numbers are useful for exact proportions, but they do not fill the number line completely. For example, there is no rational number whose square is 2.

$$\sqrt{2} \notin \mathbb{Q}, \qquad \sqrt{2} \in \mathbb{R}.$$

The real number system is needed because geometry, distance, limits, and calculus naturally produce irrational values.

Worked Example: Worked Example 2: Why $\sqrt{2}$ is not rational

Suppose $\sqrt{2} = m/n$ in lowest terms. Then

$$m^2 = 2n^2.$$

Thus m^2 is even, so m is even. Write $m = 2k$. Then

$$4k^2 = 2n^2 \quad \Rightarrow \quad n^2 = 2k^2,$$

so n is also even. This contradicts the assumption that m/n was in lowest terms. Hence $\sqrt{2}$ is irrational.

Modeling meaning: Even the distance between $(0, 0)$ and $(1, 1)$ is irrational. A coordinate model therefore requires real numbers, not only fractions.

Checkpoint: Number-system check

A route-planning model returns a shortest distance of $5\sqrt{5}$ km, a travel time of 11.2 minutes, and a need for 3.4 vans. Which quantities are continuous model outputs? Which must be converted into integer decisions?

Absolute value, distance, and intervals

Core explanation, worked reasoning, and application

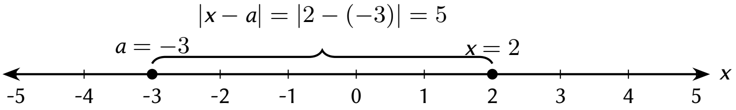
Concept: Absolute value as distance

For a real number a,

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

Geometrically, $|a|$ is the distance from a to 0 on the number line. More generally, $|x - a|$ is the distance between x and a.

Illustration: Absolute value, distance, and intervals



Absolute value measures distance on the real line.

Worked Example: Worked Example 3: Measurement tolerance

A temperature sensor is acceptable if its reading x is within $0.4°$C of the true value $25°$C. Write this condition using absolute value and interval notation.

$$|x - 25| < 0.4.$$

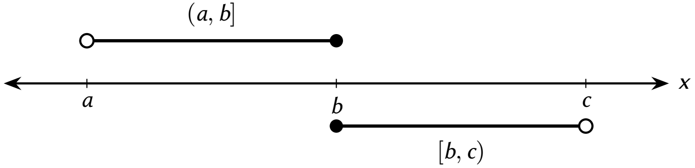
This means

$$24.6 < x < 25.4,$$

so the acceptable interval is $(24.6, 25.4)$.

Modeling interpretation: Absolute value inequalities are a compact way to state tolerance, error bounds, residual thresholds, and safety margins.

Illustration: Absolute value, distance, and intervals



Open endpoints are drawn as hollow circles; closed endpoints are filled.

Checkpoint: Interval check

Rewrite each condition in interval notation.

1. $|x - 6| \leq 2$.
2. A residual r satisfies $|r| < 0.1$.
3. A model parameter k is positive but less than 1.

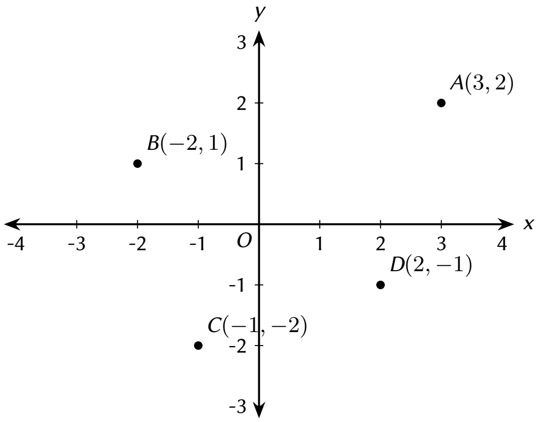
Coordinates, distance, and midpoint

Core explanation, worked reasoning, and application

Coordinates, distance, and midpoint

A point in the plane is represented by an ordered pair (x, y). In modeling, coordinates may represent a physical map, a feature space, or two variables measured from data.

Illustration: Coordinates, distance, and midpoint



Coordinates locate points relative to the *x*- and *y*-axes.

Concept: Distance and midpoint formulas

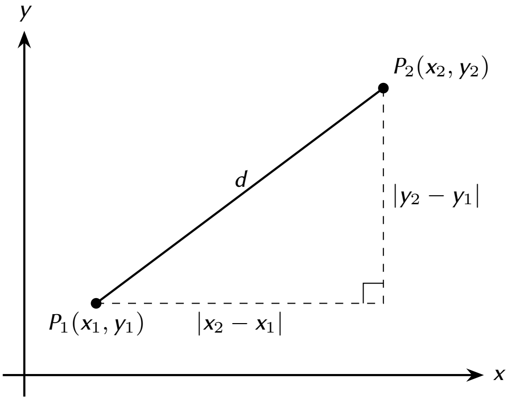
For $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The midpoint is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Illustration: Coordinates, distance, and midpoint



The distance formula is the Pythagorean theorem applied to coordinate differences.

Worked Example: Worked Example 4: Choosing a meeting point

Two students live at $A = (1, 2)$ and $B = (7, 6)$ on a simplified map measured in km. A fair meeting point halfway between them is

$$M = \left(\frac{1+7}{2}, \frac{2+6}{2}\right) = (4, 4).$$

The direct distance between them is

$$d(A, B) = \sqrt{(7-1)^2 + (6-2)^2} = \sqrt{36+16} = \sqrt{52} \approx 7.21 \text{ km}.$$

Modeling interpretation: The midpoint is fair under straight-line distance. If actual walking routes follow roads, the Euclidean midpoint may no longer be fair.

Model Critique: Model criticism: Euclidean distance versus real travel distance

The distance formula assumes a flat coordinate plane and straight-line movement. In route planning, road networks, stairs, slopes, traffic lights, rivers, gates, and restricted areas may make Euclidean distance a poor proxy. A modeler should say whether Euclidean distance is a reasonable approximation or only a first estimate.

Lines as simple models

Core explanation, worked reasoning, and application

Lines as simple models

A line is the simplest model of constant rate of change.

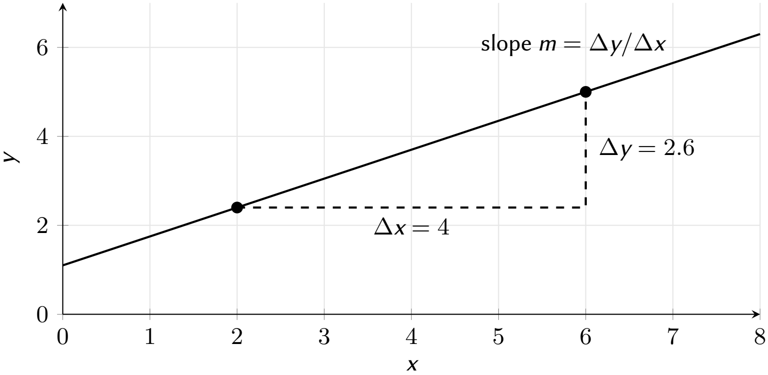
$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x}.$$

In modeling, slope carries units. If x is time in minutes and y is distance in metres, then slope is metres per minute.

Concept: Forms of a line

Form	Equation	Use
Slope-intercept	$y = mx + b$	slope and starting value are known
Point-slope	$y - y_1 = m(x - x_1)$	one point and slope are known
Two-point	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	two observations are known
General form	$Ax + By + C = 0$	useful for point-to-line distance

Illustration: Lines as simple models



The slope of a line measures constant change per unit input.

Worked Example: Worked Example 5: A linear travel-time model

A delivery robot takes 4 minutes to travel 120 m and 10 minutes to travel 300 m along a corridor. Let x be distance in metres and y be travel time in minutes. Assuming a linear model,

$$m = \frac{10-4}{300-120} = \frac{6}{180} = \frac{1}{30} \text{ min/m.}$$

Using point-slope form through $(120, 4)$,

$$y - 4 = \frac{1}{30}(x - 120),$$

so

$$y = \frac{x}{30}.$$

The intercept is 0, suggesting that the robot has negligible fixed start-up delay in this simplified case.
Interpretation: The speed is the reciprocal of the slope: 30 m/min. If later data show a nonzero intercept, that may represent loading time or waiting time.

Checkpoint: Line check

Find the line through $(2, 5)$ and $(8, 17)$. Interpret the slope if *x* is the number of hours studied and *y* is a test score.

Concept: Parallel, perpendicular, and point-to-line distance

For non-vertical lines, parallel lines have the same slope. Perpendicular lines have slopes satisfying

$$m_1 m_2 = -1.$$

The distance from (x_0, y_0) to the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

This formula measures the shortest perpendicular distance from a point to a linear boundary.

Worked Example: Worked Example 6: Distance to a safety boundary

A restricted area is approximated by the line

$$3x - 4y + 12 = 0.$$

A drone is at $P = (2, 5)$. Its shortest distance to the boundary is

$$d = \frac{|3(2) - 4(5) + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{|6 - 20 + 12|}{5} = \frac{2}{5} = 0.4.$$

If coordinates are measured in kilometres, the drone is 0.4 km from the boundary.

Circles as distance constraints

Core explanation, worked reasoning, and application

Circles as distance constraints

A circle is a set of points at a fixed distance from a centre. This makes circles useful for coverage, range, safety, and service-radius models.

Concept: Equation of a circle

A circle with centre (h, k) and radius r has equation

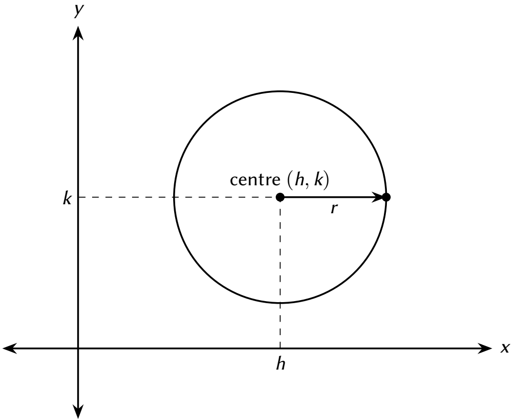
$$(x - h)^2 + (y - k)^2 = r^2.$$

The general form

$$x^2 + y^2 + Dx + Ey + F = 0$$

can be converted to standard form by completing the square.

Illustration: Circles as distance constraints



A circle models all locations within a fixed distance from a centre.

Worked Example: Worked Example 7: Coverage radius

A Wi-Fi router is located at $(3, -2)$ and has effective range 5 m. The coverage boundary is

$$(x - 3)^2 + (y + 2)^2 = 25.$$

A device at $(7, 1)$ has squared distance

$$(7 - 3)^2 + (1 + 2)^2 = 16 + 9 = 25,$$

so it lies exactly on the boundary.
Modeling interpretation: In real life, walls and interference make the boundary non-circular. The circle is a first approximation based on distance alone.

Worked Example: Worked Example 8: Completing the square

Find the centre and radius of

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

Group terms and complete the square:

$$(x^2 - 6x) + (y^2 + 4y) = 12,$$

$$(x - 3)^2 + (y + 2)^2 = 12 + 9 + 4 = 25.$$

Hence the centre is $(3, -2)$ and the radius is 5.

Activity: Differentiated geometric-modeling studio

Choose one route.

1. **Core route: facility distance.** A help desk is at $(2, 1)$. Find which of $A = (5, 5)$, $B = (-1, 3)$, and $C = (4, -2)$ is closest.
2. **Challenge route: coverage model.** A sensor at $(4, 3)$ covers radius 3. Write the coverage inequality and test whether $(6, 5)$ and $(8, 3)$ are covered.
3. **Extension route: boundary distance.** A road is modeled by $2x - y + 1 = 0$. A school at $(3, 4)$ must be at least 1 km from the road. Check whether the location is acceptable and suggest a direction to move.

Optional appendix: completeness and why calculus works

Core explanation, worked reasoning, and application

Summary: Completeness in one page

A set $S \subset \mathbb{R}$ is **bounded above** if there is a number M with $x \leq M$ for all $x \in S$. The **supremum** is the least upper bound.

The real numbers satisfy the **completeness axiom**: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

This is deeper than what is needed for ordinary coordinate geometry, but it explains why limits and calculus are possible. Rational numbers have gaps; real numbers fill those gaps.

Model Critique: Competition-style communication

When using coordinates in a report, do not merely write formulas. State:

1. what the axes represent and what units are used;
2. why Euclidean distance is acceptable, or what it ignores;
3. whether variables are continuous or must be rounded to integers;
4. how distance, slope, or circles support the final recommendation.

Weak statement: “The distance is 5.”
Stronger statement: “Using a coordinate map measured in kilometres, the Euclidean distance between the station and Building A is 5 km. This is a lower-bound estimate because actual walking routes must follow campus roads.”

Summary: Lesson summary

Concept	Modeling meaning
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$	Different number systems support counting, ratios, and continuous measurement.
Irrational number	Natural geometric quantities such as $\sqrt{2}$ cannot be expressed as fractions.
Absolute value	Measures distance from zero or error from a target.
Interval	Describes acceptable ranges, constraints, and uncertainty bands.
Coordinate plane	Converts location into computable ordered pairs.
Distance formula	Measures straight-line closeness, travel lower bounds, and error.
Slope	Constant rate of change; units matter.
Line	Simplest model of linear relationship or boundary.
Circle	Set of points at fixed distance; useful for coverage and range models.
Completeness	The hidden property of $\mathbb{R}$ that supports limits and calculus.

Checkpoint: Exit ticket

A school map uses coordinates in metres. In three sentences, explain how you would model whether a new water station covers all classrooms within 80 m. Mention the centre, the radius, and one limitation of the model.

Exercises: Core practice

1. Classify each number as natural, integer, rational, irrational, or real: $-4, 0, 7/3, \sqrt{9}, \sqrt{7}, \pi$.
2. Solve and write in interval notation: $|x - 4| < 1.5$ and $|2x + 1| \leq 5$.
3. Find the distance and midpoint between $A = (-2, 3)$ and $B = (4, -1)$.
4. Find the equation of the line through $(1, 5)$ and $(4, 11)$.
5. Find the line through $(2, -1)$ perpendicular to $y = 3x + 4$.
6. Find the centre and radius of $x^2 + y^2 + 2x - 8y + 8 = 0$.

Exercises: Modeling challenge

A campus planner wants to place a temporary information booth. Buildings are represented by points

$$A = (0, 0), \quad B = (6, 2), \quad C = (3, 7), \quad D = (-2, 4).$$

1. Compute the midpoint of A and B. Explain why this may or may not be a good booth location.
2. Find the distance from the proposed booth $P = (2, 3)$ to each building.
3. If the booth can serve students within radius 5, write the coverage circle and determine which buildings are covered.
4. A main walkway is approximated by the line $x + y = 5$. Find the distance from P to the walkway.
5. Write a short recommendation. Include one limitation of using coordinates and Euclidean distance.

Exercises: Extension practice

1. Prove that the product of a nonzero rational number and an irrational number is irrational.
2. Prove the triangle inequality $|a + b| \leq |a| + |b|$ using the fact that $-|a| \leq a \leq |a|$.
3. A set $S = (0, 1)$ has no maximum but has supremum 1. Explain the difference between maximum and supremum.
4. Find all points on the x-axis that are equidistant from $(2, 3)$ and $(-1, 4)$.
5. A circle passes through $(0, 0)$, $(4, 0)$, and $(0, 3)$. Find its equation.