Supplementary 1: Sets, Functions, and Mathematical Language

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of this supplementary lesson, readers should be able to:

1. use the language of **sets**, **elements**, **subsets**, **union**, **intersection**, and **complement**;
2. decide whether a relation is a **function** by checking whether each input has exactly one output;
3. determine the **domain** and **range** of common real functions;
4. interpret a graph using the **vertical line test**;
5. compute sums, products, quotients, and **compositions** of functions with attention to domain restrictions;
6. explain why precise mathematical language matters in mathematical modeling.

Why this supplement matters for modeling

Core explanation, worked reasoning, and application

Why this supplement matters for modeling

A mathematical model begins before any equation is written. We must decide what objects are under discussion, which values are allowed, what counts as an input, and what output is produced. These decisions are expressed using the language of sets and functions.

Activity: Launch: define the modeling universe

A school wants to model student lunch waiting time. Discuss:

1. What is the **universal set** of people under discussion: all students, all diners, or everyone on campus?
2. What input could define a function: arrival time, queue length, meal type, or payment method?
3. What output should the function return: waiting time, service time, total time, or probability of waiting more than 10 minutes?

Modeling lesson: Vague sets create vague models. A good model states the population, variables, and valid inputs explicitly.

Sets as modeling categories

Core explanation, worked reasoning, and application

Concept: Core definitions

A **set** is a collection of objects. The objects are called **elements**. If x is an element of A, write $x \in A$; if not, write $x \notin A$.

If every element of A is also in B, then A is a **subset** of B, written $A \subseteq B$. If $A \subseteq B$ but $A \neq B$, then A is a **proper subset**, written $A \subsetneq B$.

The **empty set** is written $\varnothing$. It contains no elements.

Worked Example: Worked Example 1: Classifying students using sets

Let U be the set of students in a modeling club. Define

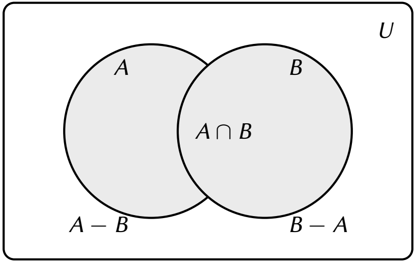
$$A = \{\text{students who know Python}\}, \qquad B = \{\text{students who have joined a modeling contest}\}.$$

Then:

- $A \cap B$ is the set of students who both know Python and have contest experience.
- $A \cup B$ is the set of students who know Python or have contest experience, or both.
- A^c is the set of students in U who do not know Python.
- $B - A$ is the set of students with contest experience but without Python experience.

These are not abstract symbols only; they are categories a teacher may use to form balanced teams.

Illustration: Sets as modeling categories



A Venn diagram is a compact way to reason about categories, overlap, and exclusions.

Concept: Set operations

Given sets A and B inside a universal set U,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}, \qquad A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}, \qquad A^c = U - A.$$

The word “or” in mathematics is usually inclusive: $x \in A \cup B$ allows x to be in both sets.

Checkpoint: Set-language check

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$, and $B = \{1, 2, 3, 5, 8\}$. Find $A \cap B$, $A \cup B$, $B - A$, and A^c. Then write one sentence interpreting each set if A means "even-numbered students" and B means "students selected for a pilot test."

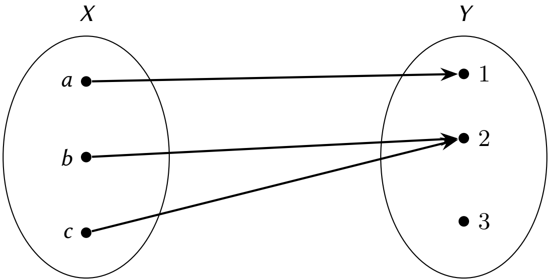
Relations and functions

Core explanation, worked reasoning, and application

Relations and functions

A **relation** from X to Y is any set of ordered pairs (x, y) with $x \in X$ and $y \in Y$. A **function** is a special relation: each input is paired with at most one output. In most calculus contexts, we require every input in the stated domain to have exactly one output.

Illustration: Relations and functions



Function: each input has exactly one output.
Different inputs may share the same output.

A many-to-one relation can still be a function. The forbidden pattern is one input producing two different outputs.

Worked Example: Worked Example 2: Is it a function?

Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

1. $R_1 = \{(1, a), (2, a), (3, b)\}$ is a function from X to Y because each input has exactly one output.
2. $R_2 = \{(1, a), (1, b), (2, c)\}$ is not a function because input 1 has two different outputs.
3. $R_3 = \{(1, a), (2, b)\}$ is a function on domain $\{1, 2\}$, but not a function with domain X if every element of X must be used.

Modeling interpretation: If the input is “student ID”, then a model for “class level” should not output two different class levels for the same student.

Concept: Domain, codomain, and range

For a function $f : X \to Y$,

- the **domain** is the input set where the function is defined;
- the **codomain** is the declared output universe Y;
- the **range** is the set of outputs that actually occur.

Symbolically,

$$\mathrm{Dom}(f) = \{x \mid f(x) \text{ is defined}\}, \qquad \mathrm{Range}(f) = \{f(x) \mid x \in \mathrm{Dom}(f)\}.$$

Real functions, domain, and graph

Core explanation, worked reasoning, and application

Real functions, domain, and graph

In calculus and modeling, a common case is a **real function**, where both inputs and outputs are real numbers. If a formula is given without a stated domain, the implied domain is the largest set of real numbers for which the formula makes sense.

Worked Example: Worked Example 3: Domain and range from a formula

Consider

$$f(x) = \sqrt{x-2}.$$

For a real-valued output, the expression inside the square root must be non-negative:

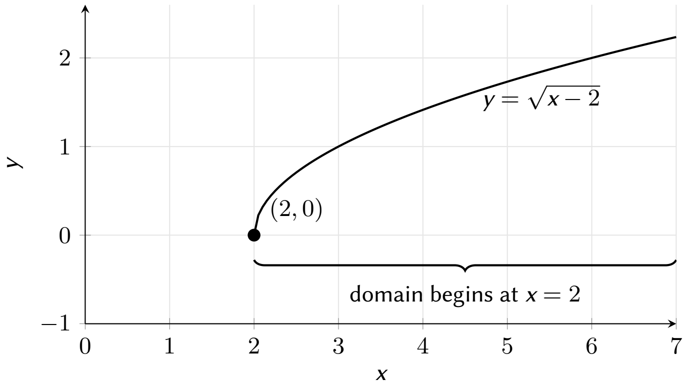
$$x - 2 \geq 0 \quad \Rightarrow \quad x \geq 2.$$

Thus

$$\mathrm{Dom}(f) = [2, \infty), \qquad \mathrm{Range}(f) = [0, \infty).$$

The starting point of the graph is $(2, 0)$.

Illustration: Real functions, domain, and graph

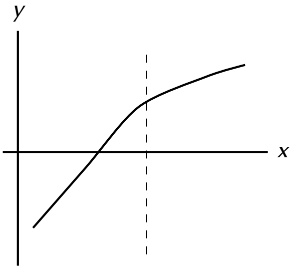


The graph of $f(x) = \sqrt{x-2}$ begins only where the formula produces real values.

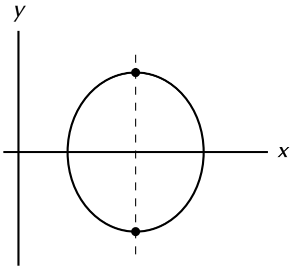
Concept: Vertical line test

A curve is the graph of a function of x if every vertical line intersects it at most once. This is the geometric version of the rule: one input, one output.

Illustration: Real functions, domain, and graph



passes



fails

The curve on the left represents a function; the circle on the right does not, because one x gives two y values.

Checkpoint: Domain check

Find the domain of each real function:

1. $f(x) = \dfrac{1}{\sqrt{x-2}}$;
2. $g(x) = \sqrt{9 - x^2}$;
3. $h(x) = \dfrac{x+1}{x^2-4}$.

For each, state the restriction in words before writing the interval or set notation.

Operations and composition

Core explanation, worked reasoning, and application

Operations and composition

Functions can be added, subtracted, multiplied, divided, and composed. These operations are useful because many models are built from submodels.

Concept: Algebra of functions

If f has domain A and g has domain B, then the sum, difference, and product are defined on $A \cap B$:

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x), \quad (fg)(x) = f(x)g(x).$$

The quotient is defined where both functions are defined and $g(x) \neq 0$:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

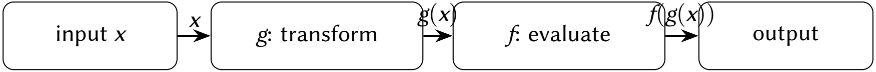
Concept: Composition

The **composite function** $f \circ g$ is

$$(f \circ g)(x) = f(g(x)).$$

The input x must first be valid for g, and then the output $g(x)$ must be valid as an input for f.

Illustration: Operations and composition



A composition is a modeling pipeline: output from one step becomes input to the next.

Composition represents a sequence of transformations.

Worked Example: Worked Example 4: Composition and domain

Let

$$f(u) = \sqrt{u}, \qquad g(x) = 4 - x^2.$$

Then

$$(f \circ g)(x) = f(4 - x^2) = \sqrt{4 - x^2}.$$

The domain is not all real numbers. Since the square root requires $4 - x^2 \geq 0$,

$$-2 \leq x \leq 2.$$

Therefore

$$\mathrm{Dom}(f \circ g) = [-2, 2].$$

Modeling interpretation: If $g(x)$ computes a predicted quantity that must be non-negative before another formula can use it, domain checking prevents meaningless outputs.

Worked Example: Worked Example 5: Iterating a function

Let

$$f(x) = \frac{1}{1+x}.$$

Compute $f(f(f(2)))$:

$$f(2) = \frac{1}{3}, \qquad f(f(2)) = f\left(\frac{1}{3}\right) = \frac{1}{1+1/3} = \frac{3}{4},$$

$$f(f(f(2))) = f\left(\frac{3}{4}\right) = \frac{1}{1+3/4} = \frac{4}{7}.$$

Iteration is important in modeling because a difference equation is just repeated function application:

$$x_{n+1} = f(x_n).$$

Model Critique: Common mistake: ignoring the domain

A formula may look algebraically correct but fail as a real function. For example,

$$\frac{1}{\sqrt{x-|x|}}$$

is never defined for any real x: if $x \geq 0$, then $x - |x| = 0$ and the denominator is zero; if $x < 0$, then $x - |x| = 2x < 0$, so the square root is not real.

Short extension: induction and binomial coefficients

Core explanation, worked reasoning, and application

Short extension: induction and binomial coefficients

This supplementary note originally contains induction and binomial coefficients. They are valuable, but in a two-hour class they should be treated as a short extension or homework pathway. They support discrete models, counting, probability, and binomial distributions later in modeling.

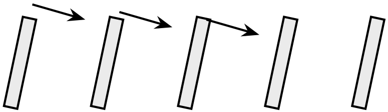
Concept: Mathematical induction

To prove a statement $P(n)$ for all positive integers n:

1. Prove the **base case**: $P(1)$ is true.
2. Prove the **inductive step**: if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for every $n \in \mathbb{Z}^+$.

Illustration: Short extension: induction and binomial coefficients



Base case starts the chain; inductive step passes truth from one case to the next.

Induction is a logical chain reaction, not a numerical pattern check.

Worked Example: Worked Example 6: Sum of the first n integers

Prove

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Base case: for $n = 1$, left side $= 1$ and right side $= 1(2)/2 = 1$.
Inductive step: assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Then

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

This is exactly the formula for $n = k+1$. Hence the formula holds for all $n \in \mathbb{Z}^+$.

Concept: Binomial coefficient

The **factorial** is

$$n! = n(n-1)\cdots 2\cdot 1, \qquad 0! = 1.$$

The **binomial coefficient**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

counts the number of ways to choose k objects from n objects. The **binomial theorem** is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Checkpoint: Extension check

1. Compute $\binom{8}{3}$ and explain what it counts.
2. Expand $(1+x)^4$ using the binomial theorem.
3. Explain why $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ by interpreting both sides as counting subsets of an n-element set.

Using these ideas in modeling reports

Core explanation, worked reasoning, and application

Model Critique: Mathematical-language checklist

When writing a model, check the following:

1. What is the universal set or population under study?
2. What is the domain of each function or variable?
3. Are there values that must be excluded, such as negative quantities or zero denominators?
4. Does each input determine exactly one output?
5. If two formulas are composed, is the output of the first formula valid as the input of the second?
6. Does the graph match the claimed function and domain?

Summary: Lesson summary

Concept	Modeling meaning
Set	Defines the population, category, or collection being modeled.
Subset	Describes a narrower category within a larger modeling universe.
Union/intersection	Combines or overlaps categories, useful for grouping and filtering data.
Complement	Represents everything in the universe not satisfying a condition.
Function	A deterministic input-output rule; each input has one output.
Domain	States which input values are valid and meaningful.
Range	States which output values the model can actually produce.
Graph	Visual representation of ordered pairs $(x, f(x))$.
Composition	Represents a sequence of model transformations.
Induction/binomial	Supports discrete reasoning, counting, and probability models.

Checkpoint: Exit ticket

A school models total cafeteria time by

$$T(n) = \sqrt{25 - n} + \frac{60}{n},$$

where n is the number of open service counters.

1. What real-number restrictions does the formula impose on n?
2. What additional real-world restrictions should be imposed?
3. Why is domain checking important before optimizing this model?

Exercises: Core practice

1. Let $U = \{1, 2, \ldots, 12\}$, $A = \{2, 4, 6, 8, 10, 12\}$, and $B = \{3, 6, 9, 12\}$. Find $A \cup B$, $A \cap B$, $A - B$, and $(A \cup B)^c$.
2. Decide whether each relation is a function: (a) $\{(1, 2), (2, 2), (3, 4)\}$; (b) $\{(1, 2), (1, 3), (2, 4)\}$; (c) $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$; (d) $\{(x, y) \in \mathbb{R}^2 : x = y^2\}$.
3. Find the domain of each real function: $f(x) = \sqrt{x + 5}$, $g(x) = 1/(x^2 - 9)$, $h(x) = \sqrt{4 - x}/(x + 1)$.
4. Let $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. Find $f \circ g$, $g \circ f$, $(f + g)(x)$, and $(fg)(x)$.
5. Use induction to prove $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Exercises: Modeling challenge

A school wants to form modeling teams. Let U be all students in the training program. Define sets for students who know Python, students who are strong in writing, students with previous contest experience, and students available on Saturday.

1. Define four sets using clear notation.
2. Write an expression for the set of students who know Python and are available on Saturday.
3. Write an expression for students who have writing strength but no previous contest experience.
4. Suppose each student is assigned exactly one role: coder, writer, analyst, or presenter. Is "student $\mapsto$ role" a function? Explain.
5. Suppose each student may have several strengths. Is "student $\mapsto$ strength" a function? Explain.
6. Write a short paragraph explaining how this set-and-function language can help a teacher form balanced teams.

Exercises: Extension practice

1. Prove by induction that
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$
2. Prove that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ using the binomial theorem and then explain it using subsets.
3. Let $f(x) = 1/(1+x)$. Find a formula for the first five iterates $f(1), f^{(2)}(1), \ldots, f^{(5)}(1)$. What pattern do you notice?
4. Find the domain of $F(x) = \sqrt{\dfrac{x-1}{x+2}}$. Give your answer in interval notation and justify it with a sign chart.