Lecture 12: Graphical and Qualitative Modeling

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of the lesson, readers should be able to:

1. use the shape of a graph as a mathematical model when exact equations are unknown;
2. identify and interpret **equilibria** as intersections of response curves;
3. reason from **slope**, **concavity**, and curve shifts to predict policy effects;
4. build a simple **discrete dynamical system** and test stability;
5. explain the economic condition $MR = MC$ and connect it to supply curves;
6. analyze a per-unit tax using supply and demand graphs;
7. write a qualitative modeling conclusion with assumptions, diagrams, and limitations.

Summary: Two-hour lesson route

0–12 min	Launch: when a graph is a model.
12–28 min	Graphical vocabulary: increasing, decreasing, concavity, intersection and shift.
28–52 min	Case study I: arms-race response curves and equilibrium.
52–68 min	Discrete arms-race update and stability check.
68–88 min	Case study II: firm behavior and $MR = MC$.
88–108 min	Gasoline tax, supply-demand shifts, and tax incidence.
108–120 min	Differentiated studio, model criticism, exit ticket, and homework briefing.

Graphs as models

Core explanation, worked reasoning, and application

Graphs as models

A model does not always have to begin with a fully specified formula. In many policy and competition problems, we know qualitative features before we know numbers. A demand curve is decreasing. A cost curve may first flatten and then steepen. A deterrence requirement may increase but with diminishing additional effect. These features are mathematical information.

A **qualitative graphical model** uses the shape of one or more functions to infer behavior. The central questions are:

- Where do curves intersect?
- Is an intersection stable or unstable?
- How does a parameter shift a curve?
- Which conclusion is robust even if the exact formula changes?

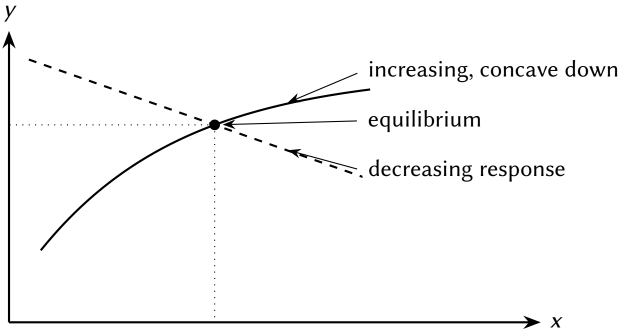
Activity: Launch: what can a graph tell us?

Consider a school canteen. Let q be the number of lunch sets prepared and let $C(q)$ be total cost.

1. Sketch a plausible cost curve. Should it be increasing? Should it be linear?
2. Sketch a plausible demand curve between price and quantity demanded.
3. State one conclusion you can make from the shapes, even without exact equations.

Modeling lesson: a graph records assumptions. A line says constant marginal effect; a curved graph says the marginal effect changes.

Illustration: Graphs as models



A graphical model turns qualitative assumptions about direction and curvature into predictions about intersections and comparative statics.

Concept: Core vocabulary

A **response curve** shows the value one part of a system requires in response to another part. An **equilibrium** is an intersection where all response requirements are simultaneously satisfied. A **comparative-static analysis** asks how the equilibrium changes after a parameter shifts a curve.

Case study I: an arms-race graphical model

Core explanation, worked reasoning, and application

Case study I: an arms-race graphical model

The arms-race model is not included because of its political content, but because it is an excellent example of qualitative modeling. The modeler does not need the exact military formula. The shape of the response curves already gives useful conclusions.
Let x be the number of weapons held by Country X and y be the number held by Country Y. Suppose

$$y = f(x)$$

means: the minimum number of Y weapons needed to maintain deterrence when X has x weapons. Similarly,

$$x = g(y)$$

means: the minimum number of X weapons needed when Y has y weapons.

Concept: Deterrence curve assumptions

For the core lesson we use three qualitative assumptions.

1. f and g are increasing: more weapons by one side require a stronger response by the other.
2. They are concave down: the first enemy weapons matter a lot; later additions may have smaller marginal impact.
3. Their intercepts represent minimum retaliation requirements even if the opponent has very few weapons.

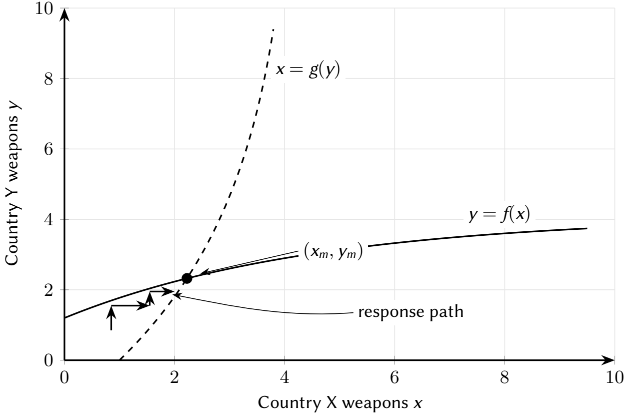
These assumptions are weaker than a full formula, but strong enough to support policy reasoning.

Worked Example: Worked Example 1: Reading the deterrence equilibrium

Suppose the two response curves intersect at (x_m, y_m). At this point, Country X has enough weapons relative to Y, and Country Y has enough weapons relative to X. The point is therefore a mutual deterrence equilibrium.

If the current state lies below $y = f(x)$, then Y judges its arsenal insufficient and builds more weapons. If the current state lies left of $x = g(y)$, then X judges its arsenal insufficient and builds more weapons. Repeated responses tend to move the system toward the intersection if each reaction is moderate.

Illustration: Case study I: an arms-race graphical model



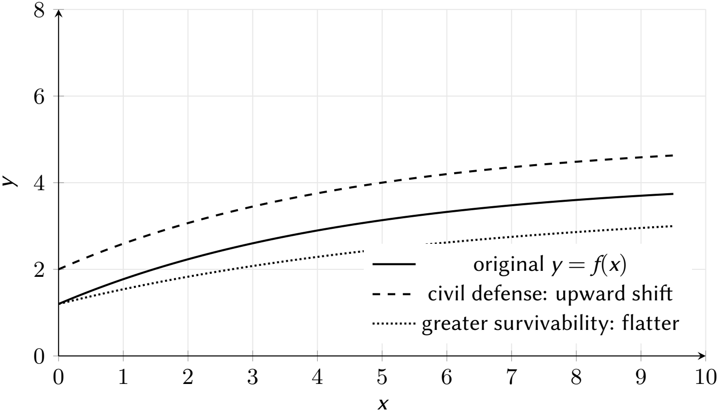
Two response curves determine a mutual deterrence equilibrium. The diagram is qualitative: the conclusions depend on shape and shifts, not exact numerical coordinates.

Worked Example: Worked Example 2: Policy shifts

The graphical model becomes useful when we ask how a policy changes a curve.

1. **Civil defense by X.** If X protects its population better, Y needs more surviving weapons to create the same deterrent effect. Y's response curve $y = f(x)$ shifts upward. The new equilibrium has a larger y, and usually a larger x as X responds to Y's buildup.
2. **Mobile missiles by X.** If X's weapons are harder to destroy, X needs fewer weapons for the same survivability. X's response curve $x = g(y)$ shifts left or becomes flatter. The equilibrium can move downward for both sides.
3. **Multiple warheads.** If one weapon can attack many targets, the exchange ratio changes. The intercept effect may reduce required launchers, but the offensive effect may steepen the response curves. A purely qualitative diagram may become ambiguous; the modeler must state that more data or a more explicit formula is needed.

Illustration: Case study I: an arms-race graphical model



Policy analysis often reduces to curve shifting. The critical question is which parameter changes and in which direction.

Checkpoint: Graphical policy check

A country builds better missile shelters, so a larger fraction of its weapons survive a first strike. Which response curve shifts: its own curve or the opponent's curve? Does the curve move upward, downward, leftward, or become flatter? Explain in two sentences.

A discrete arms-race update

Core explanation, worked reasoning, and application

A discrete arms-race update

A graph gives qualitative direction. A **discrete dynamical system** gives a sequence of numerical updates.

Worked Example: Worked Example 3: A staged arms-race model

Suppose Country Y believes it needs 120 weapons even when X has none, and then adds one additional weapon for every two weapons X has. Country X believes it needs 60 weapons even when Y has none, and then adds one additional weapon for every three weapons Y has:

$$y_{n+1} = 120 + \frac{1}{2}x_n, \qquad x_{n+1} = 60 + \frac{1}{3}y_n.$$

At equilibrium,

$$y^* = 120 + \frac{1}{2}x^*, \qquad x^* = 60 + \frac{1}{3}y^*.$$

Substitute the second equation into the first:

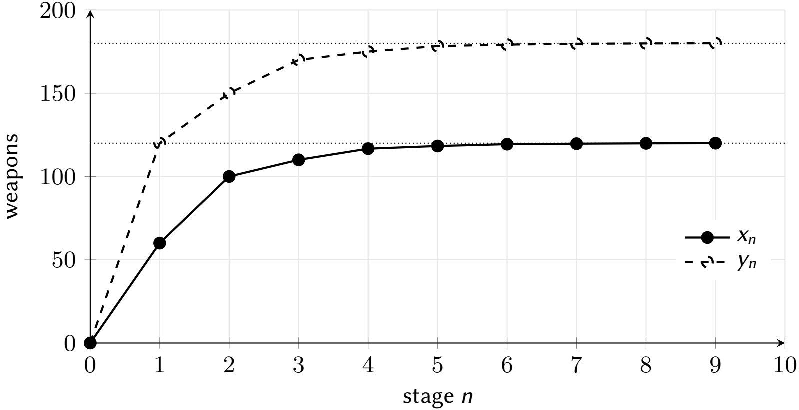
$$y^* = 120 + \frac{1}{2}\left(60 + \frac{1}{3}y^*\right) = 150 + \frac{1}{6}y^*.$$

Thus $\frac{5}{6}y^* = 150$, so

$$\boxed{y^* = 180}, \qquad \boxed{x^* = 120}.$$

The product of reaction coefficients is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} < 1$, so repeated updates converge to the equilibrium.

Illustration: A discrete arms-race update



When the product of reaction coefficients is below one, the staged arms-race model converges geometrically to equilibrium.

Model Critique: Model criticism: what does the update hide?

The update equations assume that each side reacts only to the opponent's previous stockpile, with fixed coefficients and no negotiation, budget limit, domestic politics, or uncertainty. A good modeling report should present the recurrence as a simplified warning model, not as a complete theory of political behavior.

Case study II: firm behavior and $MR = MC$

Core explanation, worked reasoning, and application

Case study II: firm behavior and $MR = MC$

Graphical modeling also explains economic decisions. Let q be output quantity. A competitive firm accepts the market price p. Its **marginal revenue** is therefore $MR = p$. Its **marginal cost** is $MC(q) = TC'(q)$.

Concept: Profit-maximizing rule

Profit is

$$\Pi(q) = TR(q) - TC(q).$$

The first-order condition is

$$\Pi'(q) = MR(q) - MC(q) = 0.$$

For a competitive firm, $MR = p$, so the firm produces where

$$\boxed{MR = MC.}$$

The firm should produce more while $MR > MC$, and stop increasing output once the next unit costs more than it earns.

Worked Example: Worked Example 4: A firm's optimal output

A small producer sells at price $p = 10$ dollars per unit and has total cost

$$TC(q) = 100 + 2q + 0.1q^2.$$

Then

$$TR(q) = 10q, \qquad \Pi(q) = 10q - (100 + 2q + 0.1q^2) = 8q - 0.1q^2 - 100.$$

The marginal cost is

$$MC(q) = TC'(q) = 2 + 0.2q.$$

Set $MR = MC$:

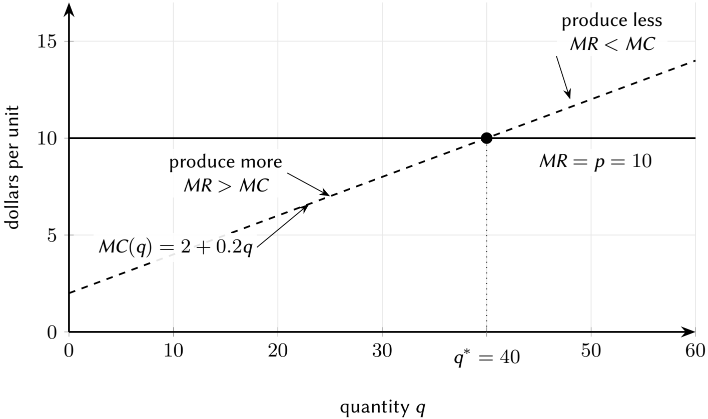
$$10 = 2 + 0.2q \quad \Rightarrow \quad q^* = 40.$$

The maximum profit is

$$\Pi(40) = 8(40) - 0.1(40)^2 - 100 = 60.$$

The graph shows the same conclusion: produce until the horizontal price line meets the marginal-cost curve.

Illustration: Case study II: firm behavior and $MR = MC$



For a competitive firm, the profit-maximizing output occurs where price equals marginal cost.

Checkpoint: Economic interpretation

If a per-unit tax of 2 dollars is imposed on the producer, the marginal cost becomes $MC(q) = 4 + 0.2q$. What is the new optimal output? Explain why the output falls.

Gasoline tax: supply, demand, and tax incidence

Core explanation, worked reasoning, and application

Gasoline tax: supply, demand, and tax incidence

Now move from one firm to a market. The **supply curve** gives the price at which producers are willing to supply quantity q. The **demand curve** gives the price at which consumers are willing to buy quantity q. The intersection is the **market equilibrium**.

A per-unit tax shifts the supply curve upward by the tax amount. The price paid by consumers rises, the quantity falls, and the tax burden is shared between consumers and producers.

Worked Example: Worked Example 5: Linear tax incidence

Suppose

$$S(q) = 1 + 0.5q, \qquad D(q) = 5 - 0.5q.$$

The original equilibrium solves

$$1 + 0.5q = 5 - 0.5q \quad \Rightarrow \quad q^* = 4, \qquad p^* = 3.$$

A per-unit tax $\tau = 1$ shifts supply to

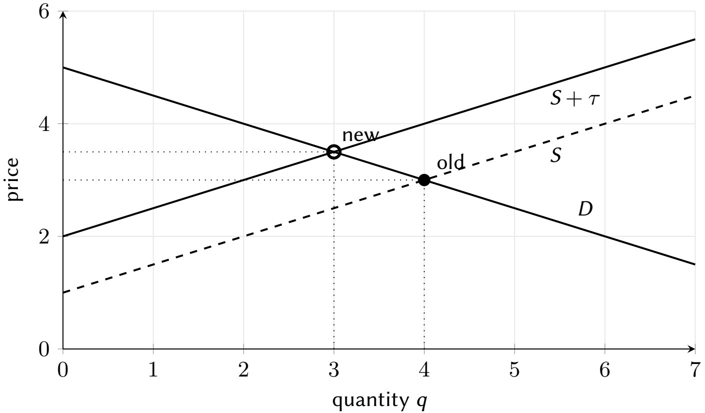
$$S_1(q) = S(q) + 1 = 2 + 0.5q.$$

The new equilibrium solves

$$2 + 0.5q = 5 - 0.5q \quad \Rightarrow \quad q_1 = 3, \qquad p_1 = 3.5.$$

Consumers pay $p_1 - p^* = 0.5$ more per unit. Producers receive $p_1 - \tau = 2.5$, which is 0.5 less than before. The one-dollar tax is split equally in this symmetric example.

Illustration: Gasoline tax: supply, demand, and tax incidence

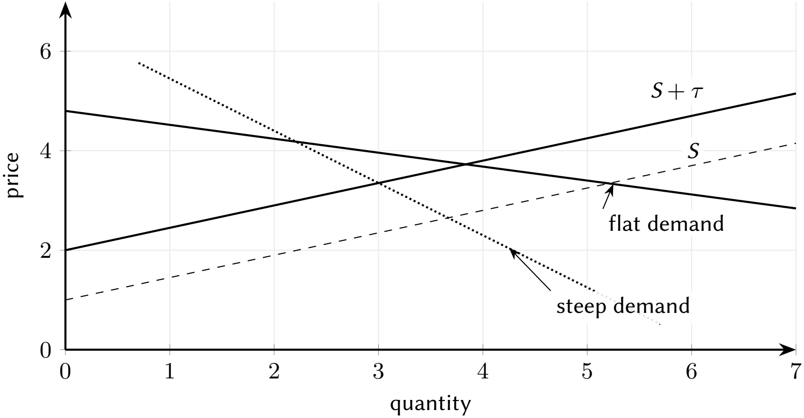


A per-unit tax shifts the supply curve upward. Price rises by less than the tax; quantity falls.

Concept: Tax incidence and elasticity

The side of the market with less flexibility bears more of the tax. If demand is steep, consumers cannot reduce quantity easily, so consumers pay most of the tax. If demand is flat, consumers can switch away easily, so producers absorb more of the tax.

Illustration: Gasoline tax: supply, demand, and tax incidence



The same tax has different effects depending on the slope of demand. This is a qualitative conclusion that does not require exact functional forms.

Model Critique: Short-run versus long-run gasoline demand

Immediately after an oil shock, demand is often inelastic: people still need to commute, cars cannot be replaced instantly, and public transport may be unavailable. A gasoline tax then raises prices but may reduce quantity only slightly. Over several years, demand becomes more elastic as people buy efficient cars, change routes, carpool, or use public transport. The same tax can then reduce consumption more effectively. The policy conclusion depends on the time horizon.

Differentiated studio: choosing a graph-based model

Core explanation, worked reasoning, and application

Activity: Differentiated modeling studio

Choose one route.

1. **Core route: tax graph.** Draw supply and demand. Shift supply upward by a tax. Label old and new equilibria. State who pays more when demand is steep.
2. **Challenge route: staged update.** For

$$y_{n+1} = 200 + \frac{1}{3}x_n, \qquad x_{n+1} = 80 + \frac{1}{4}y_n,$$

   find the equilibrium and test stability.
3. **Extension route: curve-shift argument.** Explain, using deterrence curves, why increasing survivability may reduce equilibrium weapons, while increasing offensive exchange ratio may increase them.

Summary: Choosing a graph-based modeling tool

Situation	Graphical tool	Question answered
Two sides respond to each other	Response curves	Where is mutual satisfaction?
Repeated strategic reaction	Difference equations	Does the process converge?
Firm chooses output	MR and MC curves	What quantity maximizes profit?
Market clears	Supply and demand	What price and quantity occur?
Policy intervention	Curve shift	How does equilibrium change?
Unknown exact formula	Shape and slope reasoning	Which conclusion is robust?

Writing qualitative-model conclusions

Core explanation, worked reasoning, and application

Model Critique: Competition-style communication

A qualitative graphical model should not sound vague. A strong paragraph should include:

1. **Variables:** what the axes represent.
2. **Shape assumptions:** increasing/decreasing, concavity, intercepts, or stability.
3. **Mechanism:** why those shapes are reasonable.
4. **Conclusion:** what happens to the equilibrium after a policy change.
5. **Limitation:** what cannot be decided without data.

Weak conclusion: "The tax makes price go up."
Stronger conclusion: "A per-unit gasoline tax shifts the supply curve upward. The new equilibrium has higher consumer price and lower quantity. Since short-run gasoline demand is steep, most of the tax burden is borne by consumers and quantity falls only modestly; therefore the tax is poor as an immediate conservation policy but may be more effective in the long run when demand becomes more elastic."

Summary: Lesson summary

Concept	Modeling meaning
Qualitative graphical model	Uses direction, curvature, and intersections when exact equations are unknown.
Response curve	Shows what one part of a system requires in response to another.
Equilibrium	Intersection where all requirements are satisfied simultaneously.
Curve shift	Represents a change in policy, technology, cost, or preference.
Staged arms race	Difference-equation version of response curves; stable if reaction product is below one.
$MR = MC$	Competitive firm produces until the next unit costs what it earns.
Supply-demand equilibrium	Intersection of willingness to supply and willingness to buy.
Tax incidence	Tax burden falls more on the less flexible side of the market.
Model criticism	State shape assumptions and identify when a conclusion is only qualitative.

Checkpoint: Exit ticket

In three sentences, explain why graphical modeling can be useful even when exact formulas are unknown. Use either the arms-race model or the gasoline-tax model as your example.

Exercises: Core practice

1. A response curve $y = f(x)$ shifts upward. Explain what happens to the intersection with an unchanged increasing curve $x = g(y)$, using a diagram.
2. For $y_{n+1} = 50 + 0.4x_n$, $x_{n+1} = 30 + 0.2y_n$, find the equilibrium and test stability.
3. A firm has $TC(q) = 200 + 4q + 0.05q^2$ and sells at price $p = 12$. Find q^* and maximum profit.
4. Supply and demand are $S(q) = 2 + 0.3q$, $D(q) = 8 - 0.7q$. Find equilibrium. Then impose a tax $\tau = 1$ and find the new equilibrium.
5. Draw two demand curves, one steep and one flat. For the same supply shift, explain which case gives a larger quantity reduction.

Exercises: Modeling challenge

A city wants to reduce private-car use near a school. It can either impose a parking fee, improve bus service, or redesign the school gate area. Build a qualitative graphical model.

1. Define the axes for your graph. Possible choices: cost of driving vs number of cars, waiting time vs number of students, or supply-demand for parking spaces.
2. State the shape assumptions for each curve.
3. Show how one policy shifts one curve.
4. Predict the direction of change in equilibrium.
5. Write a short recommendation and state what data would be needed to make the prediction quantitative.

Exercises: Extension practice

1. For the staged arms-race system

$$y_{n+1} = a + bx_n, \qquad x_{n+1} = c + dy_n,$$

   derive the equilibrium and show that convergence requires $|bd| < 1$.
2. For linear supply and demand $S(q) = \alpha + \beta q$, $D(q) = \gamma - \delta q$, derive the consumer share of a unit tax τ.
3. Construct a graphical argument for why a subsidy to public transport may reduce gasoline use more in the long run than in the short run.
4. Explain one limitation of the arms-race response-curve model that cannot be fixed by drawing a more detailed graph.

Summary: Optional appendix: smooth deterrence formula

One possible analytic form for a deterrence curve is

$$y = \frac{y_0}{s^{x/y}}, \qquad 0 < s < 1.$$

It says that if X can target Y's missiles more times, Y needs more weapons to retain enough survivors. The formula is less important than its qualitative properties: it is increasing and concave down. In a student report, this formula should be introduced only after the assumptions have been explained in words.