Lecture 11: Dimensional Analysis and Similitude

Mathematical Modeling Lecture

Mathematical Modeling

Kenneth Sok Kin Cheng

Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of the lesson, students should be able to:

1. explain why **dimensional analysis** reduces experimental complexity;
2. check whether an equation is **dimensionally compatible**;
3. represent physical quantities using the MLT system: **mass**, **length**, and **time**;
4. construct a **dimensionless product** by solving exponent equations;
5. apply **Buckingham's Pi Theorem** to reduce a variable list;
6. use dimensional analysis to derive scaling laws such as wind force, pendulum period, and roasting time;
7. explain **similitude** and why model experiments must match key dimensionless numbers;
8. write a modeling paragraph that states variables, assumptions, Π-groups, validation, and limitations.

Why dimensional analysis?

Core explanation, worked reasoning, and application

Why dimensional analysis?

A physical model often begins with a long list of possible variables. If an experimenter studies n variables and tests k values of each, a full grid requires k^n trials. This is the **curse of dimensionality** in experimental design. Dimensional analysis does not find the whole law by magic. Instead, it says: physical laws cannot depend on whether we measure in metres or feet, kilograms or pounds. Therefore the important relationships can be expressed using *dimensionless* combinations of variables. This can reduce n original variables to far fewer dimensionless groups.

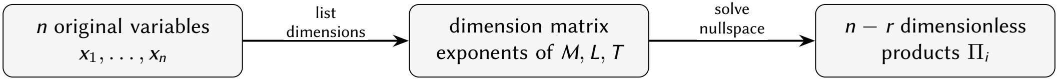
Activity: Launch: the experimental burden

12 min

A team wants to study the drag force on a small underwater robot. They suspect that drag depends on velocity v, length L, water density ρ, viscosity μ, and frontal area A.

1. If each variable is tested at 5 values, how many experiments are required in a full grid?
2. Why might this be unrealistic for a school or competition project?
3. What kind of mathematical trick would help reduce the number of experiments?

Illustration: Why dimensional analysis?



Experiments vary Π-groups, not every original variable independently.

Dimensional analysis converts a long variable list into a smaller set of dimensionless groups.

Dimensions and compatibility

Core explanation, worked reasoning, and application

Dimensions and compatibility

In mechanics, most quantities can be expressed using three primary dimensions:

$$M = \text{mass}, \qquad L = \text{length}, \qquad T = \text{time}.$$

A velocity has dimension LT^{-1}; a force has dimension MLT^{-2}; an energy has dimension ML^2T^{-2}.

Concept: Dimensional compatibility

An equation is **dimensionally compatible** if every term being added or subtracted has the same dimension. Products and powers may have different dimensions, but sums must match.
For example, in

$$s = s_0 + v_0 t - \frac{1}{2} g t^2,$$

each term has dimension L. Therefore the equation passes the compatibility check.

Dimensions and compatibility

Quantity	Dimension	Typical units
Velocity v	LT^{-1}	m/s
Acceleration a	LT^{-2}	m/s^2
Force F	MLT^{-2}	N
Density ρ	ML^{-3}	kg/m^3
Pressure p	$ML^{-1}T^{-2}$	Pa
Dynamic viscosity μ	$ML^{-1}T^{-1}$	Pa s
Energy E	ML^2T^{-2}	J
Power P	ML^2T^{-3}	W

Worked Example: Worked Example 1: catching a wrong model before collecting data

A student proposes the model

$$F = mv + v^2$$

for the force needed to move an object.

$$[F] = MLT^{-2}, \qquad [mv] = M \cdot LT^{-1} = MLT^{-1},$$

while

$$\left[v^2\right] = L^2 T^{-2}.$$

The two terms on the right cannot be added, and neither has the dimension of force. The model is structurally wrong before any numerical fitting begins.

Modeling lesson: dimensional checks are a cheap error detector. They do not prove a model is correct, but they quickly reject many impossible models.

Checkpoint: Quick compatibility check

For each equation, decide whether it is dimensionally compatible.

1. $E = \frac{1}{2}mv^2$.
2. $p = \rho g h$.
3. $F = \rho A + mv$.
4. $t = 2\pi\sqrt{L/g}$.

Constructing dimensionless products

Core explanation, worked reasoning, and application

Constructing dimensionless products

A **dimensionless product** is a product of powers of variables whose total dimension is $M^0 L^0 T^0$. Suppose variables $x_1, \ldots, x_n$ have dimensions. We search for exponents $a_1, \ldots, a_n$ such that

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

is dimensionless. Equating powers of M, L, T gives a homogeneous linear system.

Worked Example: Worked Example 2: the simple pendulum

Let the period t of a pendulum depend on length ℓ, mass m, gravity g, and release angle θ. The variables are:

$$t(T), \qquad \ell(L), \qquad m(M), \qquad g(LT^{-2}), \qquad \theta(1).$$

Consider

$$t^a \ell^b m^c g^d \theta^e.$$

Its dimension is

$$T^a L^b M^c (LT^{-2})^d = M^c L^{b+d} T^{a-2d}.$$

For this to be dimensionless,

$$c = 0, \qquad b + d = 0, \qquad a - 2d = 0.$$

There are two free choices. Taking $d = 1$ gives $a = 2$, $b = -1$, $c = 0$, so

$$\Pi_1 = \frac{gt^2}{\ell}.$$

Taking $e = 1$ gives

$$\Pi_2 = \theta.$$

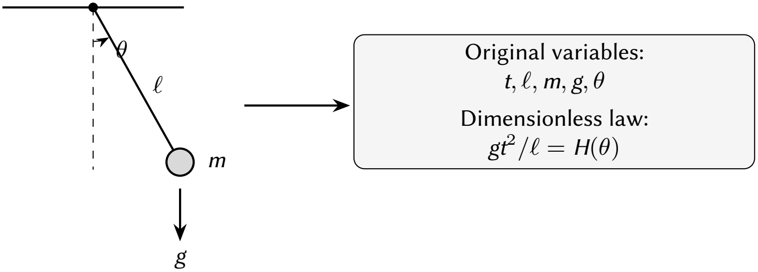
Worked Example: Worked Example 2: the simple pendulum

Therefore the period law must have the form

$$\frac{gt^2}{\ell} = H(\theta), \qquad \text{or} \qquad \boxed{t = \sqrt{\frac{\ell}{g}}\, h(\theta).}$$

Interpretation: mass disappears. Dimensional analysis predicts that pendulum period does not depend on the bob's mass.

Illustration: Constructing dimensionless products



For the pendulum, dimensional analysis removes mass and identifies the correct scale $\sqrt{\ell/g}$.

Worked Example: Worked Example 3: wind force and a missing variable

A first attempt might say wind force on a sign depends only on wind speed v and area A:

$$F = kv^a A^b.$$

But

$$\left[v^a A^b\right] = (LT^{-1})^a (L^2)^b = L^{a+2b} T^{-a},$$

which has no mass dimension. Since $[F] = MLT^{-2}$, a mass-bearing variable is missing.
Include air density ρ with dimension ML^{-3}:

$$F = k\rho^c v^a A^b.$$

Equating dimensions gives

$$M : 1 = c, \qquad T : -2 = -a \Rightarrow a = 2,$$

$$L : 1 = -3c + a + 2b = -3 + 2 + 2b \Rightarrow b = 1.$$

Thus

$$\boxed{F = k\rho v^2 A.}$$

Interpretation: doubling wind speed quadruples the force; doubling exposed area doubles the force. The dimensionless constant k must be found experimentally.

Buckingham's Pi theorem

Core explanation, worked reasoning, and application

Concept: Buckingham's Pi theorem

If a model uses n dimensional variables and the dimension matrix has rank r, then the physical relationship can be expressed using

$$n - r$$

independent dimensionless products:

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) = 0.$$

For most mechanics problems using M, L, T, the rank is often $r = 3$, so the number of groups is usually $n - 3$.

Summary: A teachable seven-step workflow

1. State the dependent variable and plausible independent variables.
2. Write the MLT dimension of every variable.
3. Build products of powers and solve for dimensionless combinations.
4. Check that every Π-group is truly dimensionless.
5. Use Buckingham's theorem to write the reduced relationship.
6. Interpret the scaling law and identify what still needs data.
7. Validate using plots of dimensionless variables, not only original variables.

Worked Example: Worked Example 4: projectile range

The horizontal range R of a projectile depends on initial speed v, gravity g, and launch angle θ.

$$R(L), \qquad v(LT^{-1}), \qquad g(LT^{-2}), \qquad \theta(1).$$

There are four variables and rank two here because only L and T appear. Therefore there are $4-2=2$ dimensionless products.
Let

$$R^a v^b g^c \theta^d.$$

Its dimension is

$$L^a(LT^{-1})^b(LT^{-2})^c = L^{a+b+c}T^{-b-2c}.$$

For dimensionless products:

$$a+b+c=0, \qquad -b-2c=0.$$

Choose $a=1$. Then $b=-2$, $c=1$, giving

$$\Pi_1 = \frac{Rg}{v^2}.$$

Worked Example: Worked Example 4: projectile range

The angle is already dimensionless, so $\Pi_2 = \theta$. Hence

$$\frac{Rg}{v^2} = H(\theta), \qquad \text{or} \qquad \boxed{R = \frac{v^2}{g} H(\theta).}$$

The exact mechanics result $R = v^2 \sin(2\theta)/g$ is consistent with the dimensional result. Dimensional analysis finds the scale v^2/g but not the particular function $\sin(2\theta)$.

Model Critique: What dimensional analysis can and cannot do

Dimensional analysis can reveal missing variables, impossible equations, scaling exponents, and useful dimensionless combinations. It cannot determine dimensionless constants or arbitrary functions such as $H(\theta)$. Those require theory, data, or simulation.

Scaling laws and model refinement

Core explanation, worked reasoning, and application

Worked Example: Worked Example 5: turkey roasting and the $w^{2/3}$ law

Suppose roasting time t depends on a characteristic length ℓ, thermal diffusivity κ with dimension L^2T^{-1}, and a dimensionless temperature ratio. The only way to combine t, ℓ, and κ dimensionlessly is

$$\Pi = \frac{\kappa t}{\ell^2}.$$

Therefore, for fixed oven conditions,

$$t \propto \frac{\ell^2}{\kappa}.$$

If turkeys are geometrically similar, weight w is proportional to volume, so

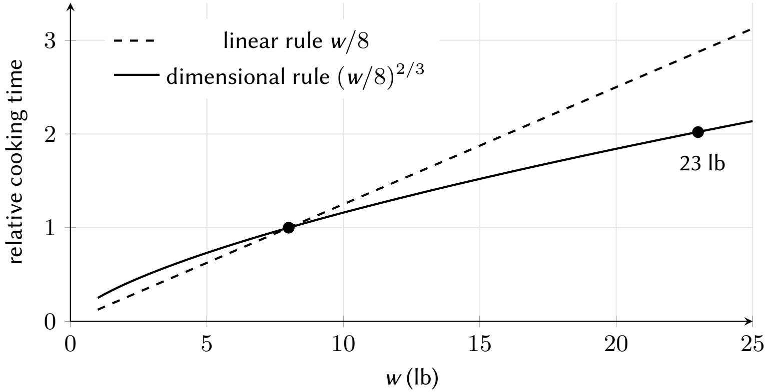
$$w \propto \ell^3, \qquad \ell \propto w^{1/3}.$$

Thus

$$\boxed{t \propto w^{2/3}.}$$

This differs from the folk rule “minutes per pound”, which implies $t \propto w$. The folk rule tends to overcook large turkeys.

Illustration: Scaling laws and model refinement



The $w^{2/3}$ law grows more slowly than a linear minutes-per-pound rule.

Worked Example: Worked Example 6: raindrop terminal velocity

Let terminal velocity v depend on radius r, gravity g, air density ρ, and dynamic viscosity μ:

$$v(LT^{-1}), \quad r(L), \quad g(LT^{-2}), \quad \rho(ML^{-3}), \quad \mu(ML^{-1}T^{-1}).$$

There are 5 variables and rank 3, so we expect 2 dimensionless products. One convenient set is

$$\Pi_1 = \frac{v}{\sqrt{rg}}, \qquad \Pi_2 = \frac{r^{3/2}\sqrt{g}\,\rho}{\mu}.$$

Therefore

$$\boxed{v = \sqrt{rg}\,H\left(\frac{r^{3/2}\sqrt{g}\,\rho}{\mu}\right).}$$

Interpretation: terminal velocity is not just proportional to $\sqrt{rg}$; viscosity and density determine the drag regime through the second dimensionless group. This is why small mist droplets, raindrops, and hailstones do not follow exactly the same simple law.

Checkpoint: Scaling interpretation

In the wind-force model $F = k\rho v^2 A$, what happens to force if:

1. wind speed triples while ρ and A are unchanged?
2. a sign is replaced by a geometrically similar sign twice as tall and twice as wide?
3. the same car drives at the same speed at high altitude where air density is lower?

Similitude: from model to prototype

Core explanation, worked reasoning, and application

Similitude: from model to prototype

Similitude means that a laboratory model and the real prototype have the same relevant dimensionless products. Geometric similarity alone is not enough. A small model submarine is not dynamically similar to a real submarine unless the key force ratios also match.

Concept: Classical dimensionless numbers

Name	Formula	Physical meaning
Reynolds number Re	$\frac{vL\rho}{\mu}$	inertia / viscosity
Froude number Fr	$\frac{v^2}{gL}$	inertia / gravity
Mach number Ma	$\frac{v}{c}$	speed / sound speed
Weber number We	$\frac{\rho v^2 L}{\sigma}$	inertia / surface tension
Drag coefficient C_D	$\frac{F_D}{\frac{1}{2}\rho v^2 A}$	dimensionless drag

Worked Example: Worked Example 7: Froude similarity for a spillway model

A 1:20 scale model of a dam spillway is tested in a lab. The prototype flow speed is $v_p = 8$ m/s. Gravity effects dominate, so match the Froude number

$$Fr = \frac{v^2}{gL}.$$

Let $L_m = L_p/20$. Froude similarity requires

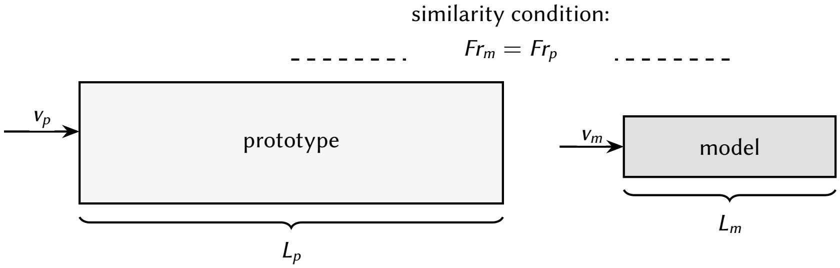
$$\frac{v_m^2}{gL_m} = \frac{v_p^2}{gL_p}.$$

Thus

$$v_m = v_p\sqrt{\frac{L_m}{L_p}} = 8\sqrt{\frac{1}{20}} \approx 1.79 \text{ m/s}.$$

Interpretation: model velocity is not $8/20 = 0.4$ m/s. For gravity-controlled free-surface flow, velocity scales with the square root of length.

Illustration: Similitude: from model to prototype



A scaled model predicts a prototype only when the relevant dimensionless numbers match.

Model Critique: Partial similitude

In real experiments, it may be impossible to match all dimensionless numbers at once. A ship model may need Froude similarity for waves and Reynolds similarity for viscosity, but the required model speeds may conflict. A strong report should state which Π-groups were matched, which were not, and how this affects confidence in the prediction.

Differentiated modeling studio

Core explanation, worked reasoning, and application

Activity: Dimensional-analysis studio

18 min

Choose one route.

1. **Core route: projectile range.** Derive $R = (v^2/g)H(\theta)$ using dimensions, then explain what dimensional analysis leaves unknown.
2. **Challenge route: Stokes drag.** Assume drag force F on a slow sphere depends on radius r, speed v, and viscosity μ. Show that $F = k\mu rv$.
3. **Extension route: blast-wave scaling.** Suppose blast radius R depends on energy E, air density ρ, and time t. Derive the scaling law $R \propto (Et^2/\rho)^{1/5}$.

Model Critique: Competition-style communication

A good dimensional-analysis paragraph should include:

1. the dependent variable and the assumed independent variables;
2. the dimension table and the number of resulting Π-groups;
3. the reduced dimensionless relationship;
4. the physical interpretation of the scaling;
5. a validation plan and a limitation.

Weak: "The answer is $t \propto w^{2/3}$."
Stronger: "Assuming geometrically similar turkeys and fixed oven conditions, the cooking time must scale as $t \propto \ell^2/\kappa$. Since weight is proportional to volume, $\ell \propto w^{1/3}$ and hence $t \propto w^{2/3}$. This predicts that large turkeys need less time per pound than small turkeys. The main limitation is that real turkeys are not perfectly similar and ovens do not heat uniformly."

Summary: Lesson summary

Concept	Modeling meaning
MLT system	Expresses physical quantities using mass, length, and time.
Compatibility	Additive terms must have the same dimension; a fast model-error check.
Dimensionless product	A product of powers whose total dimension is $M^0 L^0 T^0$.
Buckingham Π theorem	Reduces n dimensional variables to $n - r$ dimensionless groups.
Scaling law	Reveals how changing size, speed, density, or time changes outcomes.
Similitude	Model and prototype must match relevant Π-groups.
Partial similitude	Match the dominant physics; disclose unmatched effects.
Validation	Fit or compare dimensionless variables, not only raw variables.

Checkpoint: Exit ticket

In three sentences, explain why dimensional analysis is useful before collecting data. Include one example of a dimensionless product.

Practice problems

Core explanation, worked reasoning, and application

Exercises: Core practice

1. Find the dimension of pressure, power, density, viscosity, and surface tension in the MLT system.
2. Check whether $p = \rho g h$, $F = ma$, and $E = mv + gh$ are dimensionally compatible.
3. For a pendulum, verify that gt^2/ℓ is dimensionless.
4. Use dimensional analysis to show that the period of a mass-spring oscillator must have the form $t = C\sqrt{m/k}$, where k is spring stiffness with dimension MT^{-2}.
5. If $F = k\rho v^2 A$, by what factor does F change when v is doubled and A is tripled?
6. For Froude similarity, a 1:25 scale model represents a river flow with prototype speed 5 m/s. Find the model speed.

Exercises: Modeling challenge

A team wants to design a small-scale experiment for the drag force F_D on a toy car moving through air. They believe F_D depends on speed v, frontal area A, air density ρ, viscosity μ, and characteristic length L.

1. List the dimensions of all variables.
2. Predict how many dimensionless groups should appear.
3. One likely group is the Reynolds number $Re = vL\rho/\mu$. Verify it is dimensionless.
4. Propose a second dimensionless group involving F_D.
5. Explain how the team should use these groups to compare a small model car and a real car.
6. State one limitation of the experiment.

Exercises: Extension practice

1. **Projectile range.** Derive $R = (v^2/g)H(\theta)$ and verify the exact formula $R = v^2 \sin(2\theta)/g$ is consistent.
2. **Terminal velocity.** Starting with v, r, g, ρ, μ, derive two independent dimensionless products.
3. **Blast wave.** Assume blast radius R depends on energy E, air density ρ, and time t. Show that $R = C(Et^2/\rho)^{1/5}$.
4. **Choosing variables.** A student models cooking time using only turkey weight w. Explain why this variable list is incomplete, then propose a better list.
5. **Partial similitude.** Explain why matching both Reynolds and Froude numbers may be impossible in a small water-channel experiment.

Summary: Optional appendix: choosing Π-groups

Different complete sets of Π-groups are possible. For example, $Re = vL\rho/\mu$ and $1/Re$ contain the same information. A good set should be:

- dimensionless;
- independent;
- physically interpretable;
- convenient for plotting and experimentation;
- chosen so the dependent variable appears in only one group when possible.