Lecture 9: Differential Equations and Dynamic Models

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of the lesson, readers should be able to:

1. explain when a **differential equation** is more appropriate than a difference equation;
2. build first-order models from the idea "rate in minus rate out" or "rate proportional to current amount";
3. solve and interpret exponential, logistic, cooling, and simple pollution models;
4. use a **phase line** to classify **equilibria** as stable or unstable;
5. apply Euler, improved Euler/Heun, and RK4 ideas to approximate dynamic models;
6. read a basic two-population **phase plane** using nullclines;
7. communicate assumptions, parameters, validation evidence, and limitations of a dynamic model.

Summary: Two-hour lesson route

0–12 min	Launch: from discrete change to continuous rate.
12–32 min	Exponential growth and decay: build, solve, and criticize.
32–52 min	Logistic refinement and phase-line reasoning.
52–70 min	Cooling and pollution as rate-balance models.
70–98 min	Numerical solution: Euler as a baseline, improved Euler, RK4, step-size checks, and stability.
98–110 min	Differentiated studio: dose interval, lake cleanup, or predator-prey phase plane.
110–120 min	Competition-style conclusion, exit ticket, and homework briefing.

Why differential equations?

Core explanation, worked reasoning, and application

Why differential equations?

A **difference equation** updates a system at fixed steps:

$$P_{n+1} = P_n + kP_n.$$

A **differential equation** describes the instantaneous rate of change:

$$\frac{dP}{dt} = kP.$$

The second form is appropriate when the change is naturally continuous, or when the time step is so small that a continuous approximation is easier to interpret.

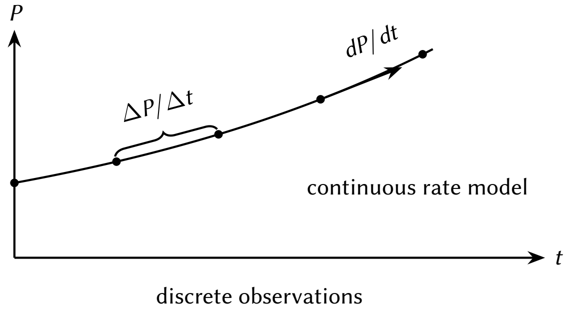
Activity: Launch: choosing the time language

For each situation, decide whether a difference equation or a differential equation is more natural. Give one reason.

1. A bank account receives interest once per month.
2. A hot drink cools continuously after being placed on a table.
3. A school records the number of absent students each morning.
4. A pollutant is continuously flushed out of a lake by a river.

Modeling lesson: The mathematics should match the timing of the mechanism, not merely the data table we happen to have.

Illustration: Why differential equations?



A differential equation idealizes many small updates as an instantaneous rate law.

Concept: Core vocabulary

A **first-order ordinary differential equation** has the form

$$\frac{dy}{dt} = f(t, y).$$

An **initial value problem** adds a starting value $y(t_0) = y_0$. A value y^* with $f(y^*) = 0$ is an **equilibrium**. If nearby solutions move toward it, it is **stable**; if they move away, it is **unstable**.

Exponential growth and decay

Core explanation, worked reasoning, and application

Exponential growth and decay

The simplest rate law says that the rate of change is proportional to the present amount:

$$\frac{dP}{dt} = kP.$$

If $k > 0$, the quantity grows exponentially; if $k < 0$, it decays exponentially.

Worked Example: Worked Example 1: Population growth and the danger of extrapolation

Suppose a population is 203.2 million in 1970 and 248.7 million in 1990. Let t be years after 1970 and assume

$$\frac{dP}{dt} = kP, \qquad P(0) = 203.2.$$

Separation gives

$$\frac{dP}{P} = k\,dt \quad \Rightarrow \quad \ln P = kt + C \quad \Rightarrow \quad P(t) = 203.2e^{kt}.$$

Use the 1990 value to estimate k:

$$248.7 = 203.2e^{20k} \quad \Rightarrow \quad k = \frac{1}{20}\ln\left(\frac{248.7}{203.2}\right) \approx 0.0101.$$

The 2000 prediction is

$$P(30) = 203.2e^{0.0101(30)} \approx 274.9 \text{ million}.$$

This is a reasonable short-term estimate, but the same model predicts unlimited growth. The equation is useful only while resources, policy, and demographics remain close to the calibration period.

Model Critique: Model criticism: what does k hide?

The parameter k compresses birth rates, death rates, immigration, age structure, economic conditions, policy, and culture into one number. A good report should say not only that $k \approx 0.0101$, but also why a constant k is a simplification and how far into the future the assumption remains credible.

Logistic growth and phase-line reasoning

Core explanation, worked reasoning, and application

Logistic growth and phase-line reasoning

Exponential growth ignores limited resources. A common refinement is the **logistic model**:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right),$$

where M is the **carrying capacity**. The factor $1 - P/M$ reduces growth as P approaches M.

Worked Example: Worked Example 2: Choosing between exponential and logistic models

A fish population in a closed pond is initially $P(0) = 80$. Ecologists believe the pond can support at most $M = 500$ fish. They estimate $r = 0.40$ per month.

$$\frac{dP}{dt} = 0.40P\left(1 - \frac{P}{500}\right).$$

The logistic solution is

$$P(t) = \frac{M}{1 + Ae^{-rt}}, \qquad A = \frac{M - P_0}{P_0}.$$

Here $A = (500 - 80)/80 = 5.25$, so

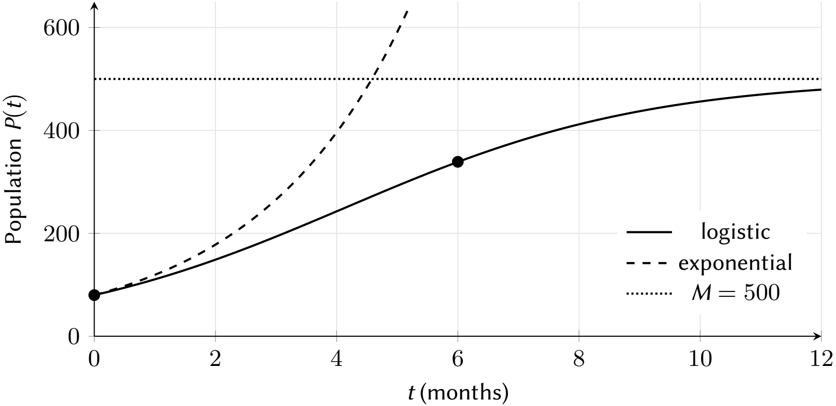
$$\boxed{P(t) = \frac{500}{1 + 5.25e^{-0.40t}}.}$$

At $t = 6$ months,

$$P(6) = \frac{500}{1 + 5.25e^{-2.4}} \approx 339.$$

The exponential model with the same initial growth rate gives $80e^{0.40(6)} \approx 882$, which already exceeds the pond's capacity. The logistic model is more credible because it encodes the physical constraint.

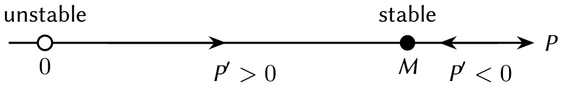
Illustration: Logistic growth and phase-line reasoning



The logistic model behaves like exponential growth at first, but bends toward the carrying capacity.

Concept: Phase line for the logistic model

For the autonomous equation $P' = rP(1 - P/M)$, equilibria occur at $P = 0$ and $P = M$. If $0 < P < M$, then $P' > 0$; if $P > M$, then $P' < 0$. Therefore 0 is unstable and M is stable.



Checkpoint: Phase-line check

For $y' = y(3 - y)$:

1. find all equilibria;
2. determine the sign of y' on each interval;
3. classify each equilibrium as stable or unstable;
4. explain the long-run behavior if $y(0) = 1$, $y(0) = 5$, and $y(0) = -1$.

Rate-balance models: cooling and pollution

Core explanation, worked reasoning, and application

Rate-balance models: cooling and pollution

Many differential equation models are built from the principle

$$\text{rate of change} = \text{rate in} - \text{rate out}.$$

This principle is easier for students to remember than a list of formula types.

Worked Example: Worked Example 3: Newton's law of cooling

A cup of tea is 90°C when placed in a room at 25°C. Newton's law of cooling assumes that the cooling rate is proportional to the temperature difference:

$$\frac{dT}{dt} = -k(T - 25), \qquad T(0) = 90.$$

The solution is

$$T(t) = 25 + 65e^{-kt}.$$

If after 10 minutes the temperature is 55°C, then

$$55 = 25 + 65e^{-10k} \quad \Rightarrow \quad e^{-10k} = \frac{30}{65} \quad \Rightarrow \quad k \approx 0.0773.$$

To find when the tea reaches 40°C:

$$40 = 25 + 65e^{-0.0773t} \quad \Rightarrow \quad t \approx 19.0 \text{ minutes}.$$

The stable equilibrium is the ambient temperature 25°C.

Worked Example: Worked Example 4: Lake cleanup as rate in minus rate out

A lake has volume $V = 10^9$ L. Clean water flows in and out at $r = 5 \times 10^7$ L/year. Let $p(t)$ be the amount of pollutant in grams. If no new pollutant enters, the pollutant leaves at concentration p/V:

$$\frac{dp}{dt} = 0 - r\frac{p}{V} = -\frac{r}{V}p = -0.05p.$$

So

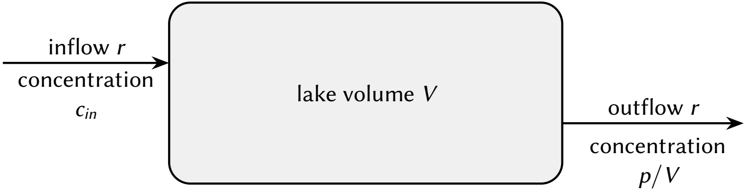
$$p(t) = p_0 e^{-0.05t}.$$

The time to reach 10% of the initial pollutant is

$$0.10p_0 = p_0 e^{-0.05t} \quad \Rightarrow \quad t = \frac{\ln 10}{0.05} \approx 46.1 \text{ years.}$$

The model explains why environmental cleanup can be slow even after pollution input stops: the relevant time scale is the **flushing time** $V/r = 20$ years.

Illustration: Rate-balance models: cooling and pollution



$$p'(t) = rc_{in} - r\,p(t)/V$$

A lake pollution model follows directly from rate in minus rate out.

Model Critique: Validation and limitations

A rate-balance model assumes perfect mixing: the pollutant concentration is the same everywhere in the lake. In a large lake, this may be false. A report should discuss whether the lake is well mixed, whether the inflow and outflow rates vary seasonally, and whether pollutant can settle into sediment and later re-enter the water.

Worked Example: Worked Example 5: Repeated drug dosage

A medicine is effective above $L = 2$ mg/L and unsafe above $H = 8$ mg/L. After a dose, the blood concentration decays according to

$$C' = -kC, \qquad k = 0.18 \text{ hr}^{-1}.$$

If each dose raises concentration by $H - L = 6$ mg/L, the residual concentration just before the next dose should approach L. For repeated dosing at interval T, the limiting residual is

$$R = \frac{H - L}{e^{kT} - 1}.$$

Setting $R = L$ gives

$$L = \frac{H - L}{e^{kT} - 1} \quad \Rightarrow \quad e^{kT} = \frac{H}{L} \quad \Rightarrow \quad \boxed{T = \frac{1}{k} \ln\left(\frac{H}{L}\right)}.$$

Here

$$T = \frac{1}{0.18} \ln 4 \approx 7.70 \text{ hours}.$$

Interpretation: an 8-hour schedule is close but slightly slower than the ideal schedule; it may be acceptable only if clinical safety margins and patient variation are considered.

Checkpoint: Dosage interpretation

If k increases because the body eliminates the drug faster, should the recommended interval T increase or decrease? Explain using the formula and in ordinary language.

Numerical solution: from Euler to reliable computation

Core explanation, worked reasoning, and application

Numerical solution: from Euler to reliable computation

Analytic formulas are useful, but many modeling ODEs cannot be solved neatly. In competitions and applied projects, it is often more realistic to state the differential equation clearly and then compute the solution numerically. The issue is not merely “press a button”. Students must understand what the numerical method is doing, how accurate it is likely to be, and how to check whether the output is trustworthy.

Concept: Numerical solution as a modeling tool

A **numerical solution** approximates the values of an unknown solution at a grid of times

$$t_0, t_1 = t_0 + h, t_2 = t_0 + 2h, \ldots$$

where h is the **step size**. A numerical method turns the rate law

$$y' = f(t, y)$$

into an update rule for y_{n+1} from information near (t_n, y_n). The smaller the step size, the more often the model is updated, but the more computation is required.

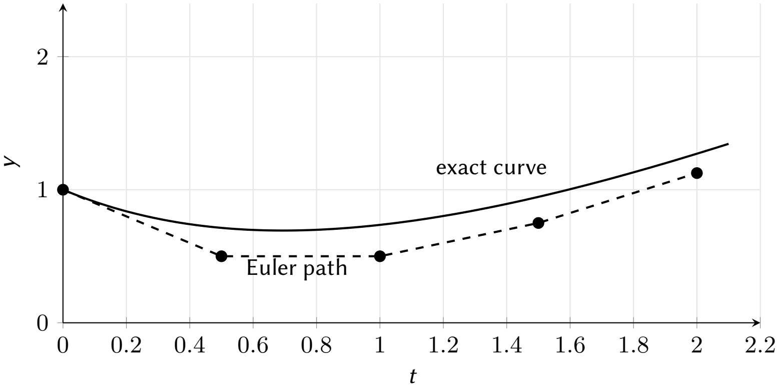
Euler's method as the baseline

Euler's method follows the tangent line at the current point:

$$\boxed{t_{n+1} = t_n + h, \qquad y_{n+1} = y_n + hf(t_n, y_n).}$$

It is valuable because it reveals the mechanism of numerical integration. However, it is only first-order accurate: the **global error** is usually $O(h)$, so halving h often roughly halves the error.

Illustration: Euler's method as the baseline



Euler's method replaces a curved solution by short tangent-line steps.

Worked Example: Worked Example 6: Euler approximation and step-size check

Approximate the solution of

$$y' = t - y, \qquad y(0) = 1,$$

on $0 \leq t \leq 2$ using $h = 0.5$.

$$y_{n+1} = y_n + 0.5(t_n - y_n).$$

n	t_n	y_n	slope $t_n - y_n$	y_{n+1}
0	0.0	1.000	-1.000	0.500
1	0.5	0.500	0.000	0.500
2	1.0	0.500	0.500	0.750
3	1.5	0.750	0.750	1.125

The exact solution is $y = t - 1 + 2e^{-t}$. At $t = 2$, exact $y(2) \approx 1.271$, so Euler gives error about 0.146. **Step-size check.** If we repeat Euler with $h = 0.25$, the estimate at $t = 2$ is approximately 1.200, with error about 0.070. The error is roughly halved, which is consistent with first-order convergence.

Checkpoint: Numerical method check

Why is a single Euler computation not enough evidence? State two checks a modeler should perform before trusting a numerical ODE result.

Improved Euler / Heun's method

Euler uses only the slope at the beginning of the step. If the slope changes across the interval, this can be crude. The **improved Euler method**, also called **Heun's method**, first predicts a trial endpoint, then averages the starting and ending slopes:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ y_{\text{pred}} &= y_n + hk_1, \\ k_2 &= f(t_n + h, y_{\text{pred}}), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2). \end{aligned}$$

This is a second-order method: its global error is typically $O(h^2)$.

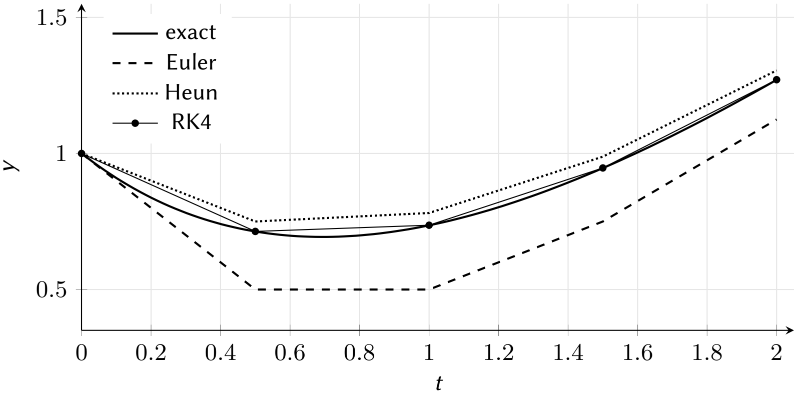
Worked Example: Worked Example 7: Euler versus Heun versus RK4

Use the same IVP $y' = t - y$, $y(0) = 1$, with $h = 0.5$ up to $t = 2$. The exact value is $y(2) = 1.27067$.

Method	Update idea	Estimate at $t = 2$	Error
Euler	starting slope only	1.12500	0.14567
Improved Euler / Heun	average of start and predicted-end slopes	1.30518	0.03451
RK4	weighted four-slope average	1.27110	0.00043

The main lesson is not that one method is always best, but that numerical method choice matters. Euler is excellent for explaining the idea; Heun and RK4 are better default tools for producing values in a report.

Illustration: Improved Euler / Heun's method



Different numerical methods may produce visibly different paths even with the same step size.

Fourth-order Runge-Kutta: the practical default

The classical **fourth-order Runge-Kutta method** balances accuracy and simplicity. It samples four slopes inside one step:

$$
\begin{aligned}
k_1 &= f(t_n, y_n),\\
k_2 &= f\!\left(t_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1\right),\\
k_3 &= f\!\left(t_n + \frac{h}{2},\ y_n + \frac{h}{2}k_2\right),\\
k_4 &= f(t_n + h,\ y_n + hk_3),\\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}
$$

RK4 has global error $O(h^4)$ for sufficiently smooth problems. In student modeling reports, it is often enough to state that the ODE was solved by RK4 or by a standard adaptive solver, and then report a step-size or tolerance check.

Concept: A numerical ODE workflow for modeling reports

1. **Write the rate law clearly.** Define variables, units, parameters, and initial conditions.
2. **Choose a method.** Use Euler for classroom illustration; use Heun, RK4, or an adaptive solver for reported numerical results.
3. **Check convergence.** Repeat with h and $h/2$, or with a tighter solver tolerance. The conclusion should not change materially.
4. **Check against known behavior.** The numerical path should respect equilibria, signs, conservation, non-negativity, or carrying capacities.
5. **Separate numerical error from modeling error.** A better solver cannot fix a wrong mechanism.

Stability of a numerical method

Accuracy is not the only issue. A method can be unstable even for a simple stable differential equation. Consider

$$y' = -10y, \qquad y(0) = 1.$$

The true solution $y = e^{-10t}$ decays smoothly to 0. Euler's method gives

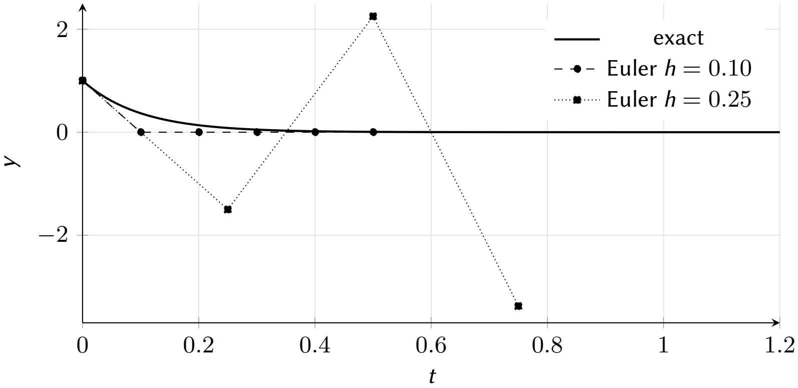
$$y_{n+1} = y_n + h(-10y_n) = (1 - 10h)y_n.$$

For the numerical solution to decay rather than explode, we need

$$|1 - 10h| < 1 \quad \Rightarrow \quad 0 < h < 0.2.$$

Thus a large step size may create artificial oscillation or blow-up even though the true model is stable.

Illustration: Stability of a numerical method



For a stable decay equation, too large a step size can create artificial numerical instability.

Model Critique: What to write in a competition report

Avoid writing only "we used Euler's method". A stronger statement is:

> *We solved the ODE numerically using RK4 with step size* $h = 0.05$*. Repeating the computation with* $h = 0.025$ *changed the predicted peak by less than 1%, so numerical error is small relative to parameter uncertainty. The model preserves non-negative populations and approaches the expected stable equilibrium.*

This sentence shows method, convergence, qualitative validation, and limitation.

Systems and phase planes: predator and prey

Core explanation, worked reasoning, and application

Systems and phase planes: predator and prey

For two interacting quantities, a single phase line is not enough. We use a **phase plane**. The key visual tools are **nullclines**: curves where one derivative is zero.

Worked Example: Worked Example 8: Predator-prey nullclines

Let $x(t)$ be prey and $y(t)$ be predator. A simple Lotka-Volterra model is

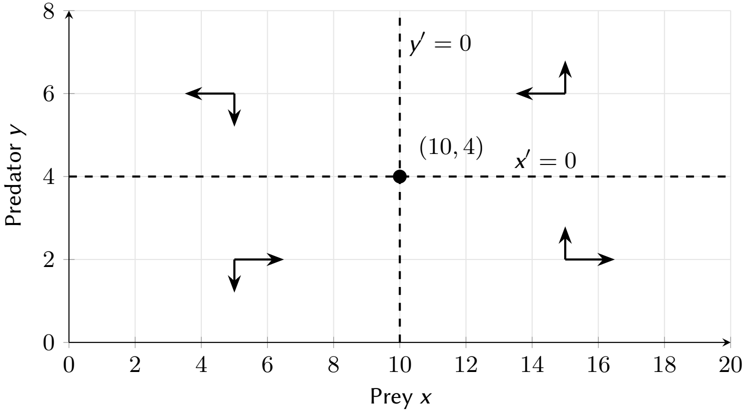
$$x' = x(2 - 0.5y), \qquad y' = y(-1 + 0.1x).$$

The x-nullclines satisfy $x' = 0$, so $x = 0$ or $y = 4$. The y-nullclines satisfy $y' = 0$, so $y = 0$ or $x = 10$. The positive equilibrium is

$$(x^*, y^*) = (10, 4).$$

If $y < 4$, prey increase; if $y > 4$, prey decrease. If $x > 10$, predators increase; if $x < 10$, predators decrease.

Illustration: Systems and phase planes: predator and prey



Nullclines divide the predator-prey phase plane into regions where each population increases or decreases.

Activity: Differentiated modeling studio

Choose one route.

1. **Core route: dosing interval.** A medicine has $L = 2$, $H = 8$, and $k = 0.18\ \text{hr}^{-1}$. Use $T = (1/k)\ln(H/L)$ and explain the result in plain language.
2. **Challenge route: lake cleanup.** A lake has $V = 3 \times 10^9$ L and $r = 10^8$ L/year. Find the time to reduce pollutant to 25% of its initial amount, then discuss one assumption that may fail.
3. **Extension route: predator-prey.** For $x' = x(1 - y)$ and $y' = y(-2 + x)$, find the nullclines, the positive equilibrium, and the signs of x' and y' in the four regions.

Summary: Choosing a differential-equation method

Situation	Recommended method	Modeling question answered
$y' = ky$ or $y' = -ky$	separation / known formula	How fast does the amount grow or decay?
$y' = rP(1 - P/M)$	logistic formula + phase line	What long-run capacity is approached?
$y' = f(y)$	phase line	Which equilibria are stable?
$y' + P(t)y = Q(t)$	integrating factor	What is the balance between input and removal?
No clean formula, classroom calculation	Euler	What does one tangent-line approximation do?
No clean formula, report-quality computation	Heun / RK4 / adaptive solver	Is the numerical prediction stable under step-size refinement?
Two interacting variables	nullclines / phase plane	Which population increases or decreases in each region?

Model Critique: Numerical error versus modeling error

If Euler's method gives a poor estimate, there are two possible causes. **Numerical error** means the step size is too large for the differential equation. **Modeling error** means the differential equation itself is not a good representation of the real system. Reducing h helps only with numerical error; it cannot fix a bad assumption such as constant temperature, perfect mixing, or unlimited growth.

Writing dynamic-model conclusions

Core explanation, worked reasoning, and application

Model Critique: Competition-style communication

A dynamic-model paragraph should contain four parts:

1. **Mechanism:** what rate law or balance law was assumed?
2. **Calibration:** how were parameters such as k, r, M, or V/r obtained?
3. **Prediction and interpretation:** what does the model imply in context?
4. **Credibility check:** what evidence supports the model, and where does it likely fail?

Weak conclusion: "The logistic model predicts 339 fish."
Stronger conclusion: "Using a carrying capacity of 500 fish and an estimated monthly growth rate of 0.40, the logistic model predicts about 339 fish after six months. The exponential model exceeds the pond capacity over the same period, so it is unsuitable for long-range prediction. The main uncertainty is the carrying capacity estimate, which should be checked against pond area, food supply, and historical population counts."

Summary: Lesson summary

Concept	Modeling meaning
Differential equation	Relates a quantity to its instantaneous rate of change.
Initial condition	Selects one solution curve from a family.
Exponential model	Rate proportional to amount; useful locally, dangerous for long extrapolation.
Logistic model	Growth slows near carrying capacity; stable equilibrium at M.
Phase line	Visual tool for one-variable autonomous DEs; shows stability.
Rate-balance model	rate in − rate out; used for pollution, mixing, and flow problems.
Euler's method	First-order tangent-line approximation; useful as a baseline but often too crude for final results.
Heun / RK4	Higher-order numerical solvers; compare step sizes before reporting quantitative predictions.
Phase plane	Visual tool for systems; nullclines locate equilibria and direction regions.
Model criticism	State assumptions, parameter sources, validation evidence, and limits.

Checkpoint: Exit ticket

In three sentences, explain the difference between an exponential model and a logistic model. Then give one real-world situation where the logistic model is more credible.

1. Solve $y' = 0.3y$, $y(0) = 12$. Find $y(5)$ and interpret the parameter 0.3.
2. A chemical decays according to $C' = -0.12C$, $C(0) = 40$. When does it fall below 5?
3. For $P' = 0.2P(1 - P/1000)$ and $P(0) = 100$, write the logistic solution and estimate $P(10)$.
4. Draw the phase line for $y' = y(y - 4)$ and classify the equilibria.
5. A room is kept at 22°C. An object cools from 80°C according to $T' = -0.05(T - 22)$. Find $T(t)$ and $T(20)$.
6. Apply Euler's method with $h = 1$ to $y' = 2 - y$, $y(0) = 0$ for three steps. Then repeat with Heun's method and compare.

Exercises: Modeling challenge

A small reservoir initially contains 12 million litres of water with pollutant concentration 0.08 g/L. Clean water enters and exits at 0.6 million litres per day. A nearby factory may continue to add pollutant at an unknown constant rate s grams/day.

1. Define $p(t)$ and write a differential equation for pollutant amount.
2. Solve the model when $s = 0$.
3. Find the long-run pollutant level when $s > 0$.
4. Suppose measurements after 10 days show concentration 0.052 g/L. Estimate whether s is closer to 0 or significantly positive.
5. Write a short recommendation for environmental officers. Include limitations of the model.

Exercises: Extension practice

1. Use separation to solve $y' = ty^2$, $y(0) = 1$. Identify whether the solution exists for all $t > 0$.
2. For $y' = y(1 - y)(y - 3)$, draw the phase line and classify all equilibria.
3. For $x' = x(3 - y)$, $y' = y(-2 + x)$, find the positive equilibrium and the nullclines. Give the direction of motion in each region.
4. Explain why Euler's method may become inaccurate or unstable if h is too large, even when the differential equation is correct. Use $y' = -10y$ as an example.

Summary: Optional appendix: integrating factor method

A first-order linear equation has standard form

$$y' + P(t)y = Q(t).$$

The integrating factor is

$$\mu(t) = e^{\int P(t)\,dt}.$$

Multiplying the equation by $\mu(t)$ turns the left side into a product derivative:

$$(\mu y)' = \mu Q.$$

Thus

$$\mu(t)y(t) = \int \mu(t)Q(t)\,dt + C.$$

This method is useful for pollution, mixing, heating/cooling with input, and many first-order engineering models.

Exercises: Additional problem bank

1. **Coffee cooling.** Coffee cools from 85°C to 60°C in 12 minutes in a 24°C room. Estimate k and predict when it reaches 45°C.
2. **Choosing a model.** A population grows from 100 to 180 to 250 to 310 over four equal time intervals, and experts believe the maximum is about 400. Explain why an exponential model may overpredict and propose a logistic model structure.
3. **Parameter sensitivity.** For the lake cleanup model $p(t) = p_0 e^{-rt/V}$, explain how the cleanup time changes if the inflow-outflow rate r doubles.
4. **Euler design.** A student uses $h = 5$ days to simulate a fast-changing epidemic model. What checks should be performed before trusting the output?
5. **Model comparison.** Write two sentences comparing a phase-line conclusion and a numerical simulation conclusion. What can each reveal that the other may hide?