Lecture 8: Game Theory and Strategic Modeling

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch	Build	Test
Question Motivating context, key vocabulary, and the first modelling decision.	**Method** Worked example, diagram or table, calculation, and parameter meaning.	**Critique** Checkpoint practice, limitation check, and communication of results.

Columns are used where comparison helps; explanation slides stay clean and spacious.

Learning Goals

By the end of the lesson, readers should be able to:

1. explain why strategic models differ from ordinary decision models;
2. identify players, strategies, payoffs, and assumptions in a game;
3. find a pure-strategy solution using **maximin**, **minimax**, and movement diagrams;
4. solve a 2×2 zero-sum game without a saddle point using mixed strategies;
5. interpret a game value as a guaranteed long-run payoff, not as a promise in one play;
6. analyse a partial-conflict game using **Nash equilibrium** and explain the Prisoner's Dilemma;
7. decide when game theory is an appropriate modeling tool and state its limitations.

Why game theory?

Core explanation, worked reasoning, and application

Why game theory?

In Lecture 7, a decision maker faced uncertain states of nature: weather, market demand, or a research report. Nature did not try to outsmart the decision maker. In **game theory**, at least one other participant is also making a decision, and that participant may change strategy after thinking about what we will do.
A **game** is a mathematical model of strategic interaction. It must specify:

- the **players**;
- each player's possible **strategies**;
- the **payoff** produced by every combination of strategies;
- what each player knows when making a decision.

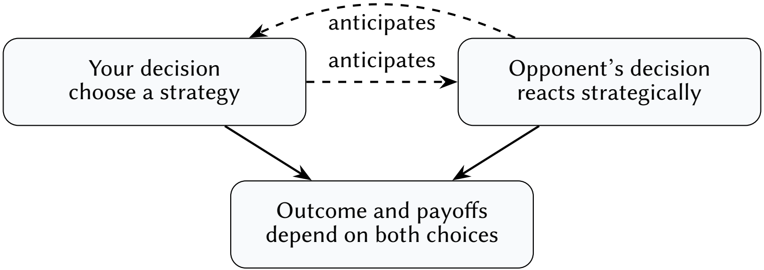
Activity: Launch: ordinary decision or game?

For each situation, decide whether a decision-theory model or a game-theory model is more appropriate.

1. A shop chooses how many umbrellas to order before the rainy season.
2. Two restaurants decide whether to offer lunch discounts.
3. A goalkeeper chooses which side to dive during a penalty kick.
4. A school decides whether to hire one or two extra canteen staff.

For the game situations, identify the players and their possible strategies.

Illustration: Why game theory?



A game is different from an ordinary decision because the other side is also reasoning.

Concept: Total conflict and partial conflict

A **total-conflict game**, often called a **zero-sum game**, is one in which one player's gain is exactly the other player's loss. A penalty kick, rock-paper-scissors, or a search-versus-hide problem is often modeled this way.

A **partial-conflict game** allows both players to gain or lose together. Advertising wars, arms races, pricing decisions, and environmental agreements often have this structure.

Pure strategies in total-conflict games

Core explanation, worked reasoning, and application

Pure strategies in total-conflict games

In a two-player zero-sum game, we write a payoff matrix from the row player's point of view. The **row player** wants larger numbers; the **column player** wants smaller numbers.

Concept: Pure strategies, maximin, minimax

A **pure strategy** is a fixed choice, such as always choosing Row 1. For a payoff matrix $A = (a_{ij})$:

$$\text{row player's maximin} = \max_i \min_j a_{ij}, \qquad \text{column player's minimax} = \min_j \max_i a_{ij}.$$

If these two numbers are equal, the game has a **saddle point**. The corresponding cell is a stable pure-strategy solution: neither player can improve by switching alone.

Worked Example: Worked Example 1: Location competition with a saddle point

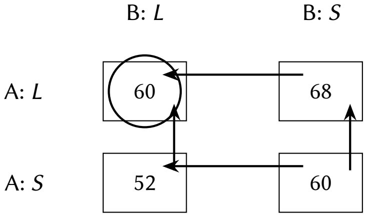
Two competing stores choose either a large district L or a small district S. The entry gives Store A's market share percentage.

Store A $\setminus$ Store B	**B: L**	**B: S**	**Row minimum**
A: L	60	68	60
A: S	52	60	52
Column maximum	60	68	

The row player guarantees at least 60 by choosing L. The column player can hold A to at most 60 by choosing L. Hence

$$\max\{60, 52\} = 60 = \min\{60, 68\}.$$

There is a saddle point at (L, L) with game value $V = 60$. The conclusion is not merely "both choose L"; it is: A can guarantee 60% market share, and B can prevent A from exceeding 60%.



arrows point into
the saddle point

Movement diagram for a pure-strategy saddle point. Vertical arrows show A's preference; horizontal arrows show B's preference.

Checkpoint: Pure strategy check

For the matrix below, find the row minima, column maxima, and decide whether a saddle point exists.

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Explain the result in words, not only by calculation.

Worked Example: Worked Example 2: Dominated strategies

Before solving a game, look for strategies that a rational player should never use. A row is **dominated** if another row gives the row player at least as much payoff in every column and strictly more in at least one column. A column is dominated if another column gives the row player at most as much payoff in every row. Consider the row player's payoff matrix:

	C_1	C_2	C_3
R_1	4	2	5
R_2	3	1	4
R_3	1	6	2

Row R_1 dominates R_2 because $4 > 3$, $2 > 1$, and $5 > 4$. Thus the row player should never choose R_2. After removing it, compare columns C_1, C_2, C_3 from the column player's viewpoint. Since the column player wants smaller payoffs, a column with larger entries is unattractive.

Dominance does not always solve the game, but it simplifies the model and often reveals strategic structure before calculation.

Model Critique: Do not remove strategies too quickly

A dominated strategy can be removed only when dominance is clear across every possible response of the opponent. Removing a strategy because it is “usually bad” is not valid. In modeling reports, state the dominance comparison explicitly.

When no pure strategy is stable: mixed strategies

Core explanation, worked reasoning, and application

When no pure strategy is stable: mixed strategies

If there is no saddle point, any fixed strategy can be exploited. The solution is to randomize. A **mixed strategy** assigns probabilities to pure strategies.

The central idea is not “being unpredictable” in a vague sense. The mathematical purpose is to make the opponent indifferent between their pure strategies, so that they cannot exploit a predictable bias.

Worked Example: Worked Example 3: Batter versus pitcher

A batter guesses Fastball F or Curve C. The pitcher throws Fastball or Curve. The entries are batting averages.

Batter \ Pitcher	Pitcher: F	Pitcher: C	Row min
Guess F	.400	.200	.200
Guess C	.100	.300	.100
Column max	.400	.300	

The maximin is .200, while the minimax is .300. Since they are unequal, no pure-strategy saddle point exists. Let x be the probability that the batter guesses Fastball. Then the batter's expected average if the pitcher throws Fastball is

$$E_F(x) = .400x + .100(1-x) = .100 + .300x.$$

If the pitcher throws Curve,

$$E_C(x) = .200x + .300(1-x) = .300 - .100x.$$

The batter wants to maximize the lower of these two lines. The best choice occurs where the two lines meet:

$$.100 + .300x = .300 - .100x \quad \Rightarrow \quad x = 0.5.$$

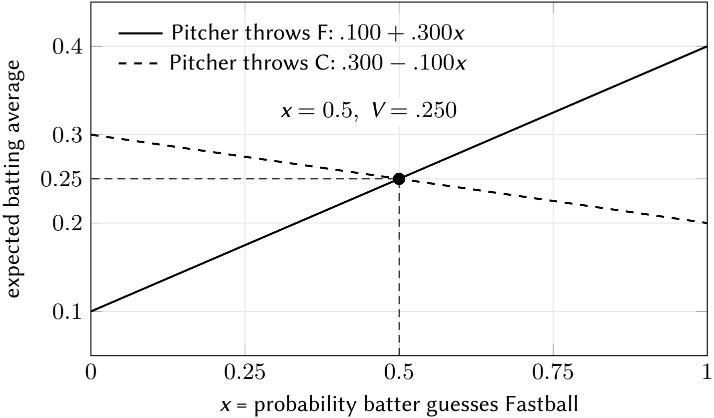
Worked Example: Worked Example 3: Batter versus pitcher

The game value is

$$V = .100 + .300(0.5) = .250.$$

So the batter should guess Fastball with probability 0.5 and Curve with probability 0.5. This guarantees a batting average of $.250$ in the long run.

Illustration: When no pure strategy is stable: mixed strategies



The batter chooses the mixture that maximizes the lower envelope of the two opponent-response lines.

Concept: Solving a 2×2 zero-sum game by indifference

For a payoff matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

let x be the probability that the row player chooses Row 1. To find the row player's optimal mix, equate the column player's two pure responses:

$$ax + c(1 - x) = bx + d(1 - x).$$

To find the column player's optimal mix, let y be the probability that the column player chooses Column 1, and equate the row player's two pure responses:

$$ay + b(1 - y) = cy + d(1 - y).$$

This method works after checking that no saddle point exists.

Worked Example: Worked Example 4: Predator and prey

A prey animal chooses Hide or Run. A predator chooses Ambush or Pursue. Entries are the prey's survival probabilities.

Prey \ Predator	Ambush	Pursue
Hide	0.30	0.60
Run	0.50	0.40

There is no saddle point. Let x be the probability the prey Hides. If the predator Ambushes,

$$E_A(x) = 0.30x + 0.50(1 - x) = 0.50 - 0.20x.$$

If the predator Pursues,

$$E_P(x) = 0.60x + 0.40(1 - x) = 0.40 + 0.20x.$$

Equating them gives

$$0.50 - 0.20x = 0.40 + 0.20x \quad \Rightarrow \quad x = 0.25.$$

The prey should Hide 25% and Run 75%. The guaranteed survival probability is

$$V = 0.50 - 0.20(0.25) = 0.45.$$

Worked Example: Worked Example 4: Predator and prey

For the predator, let y be the probability of Ambush. Equating the prey's two rows:

$$0.30y + 0.60(1-y) = 0.50y + 0.40(1-y) \quad \Rightarrow \quad y = 0.5.$$

The predator should Ambush 50% and Pursue 50%.

Model Critique: What does a mixed strategy mean?

A mixed strategy does not mean that the player is confused. It means that the player deliberately randomizes to prevent exploitation. The result is meaningful when the interaction is repeated many times, or when unpredictability itself has strategic value. In a single high-stakes play, a player may still choose a pure action based on information not represented in the matrix.

Games against nature: conservative guarantees

Core explanation, worked reasoning, and application

Games against nature: conservative guarantees

Sometimes the “opponent” is not intelligent: market demand, economy, weather, or machine failure. If probabilities are known, Lecture 7 expected value is appropriate. If probabilities are unknown and the decision maker wants a guaranteed floor, we may treat nature as adversarial.

Worked Example: Worked Example 5: Small or large production

A firm chooses Small or Large production. The economy may be Poor or Good. Payoffs are in thousands of dollars.

Firm \ Economy	**Poor**	**Good**	**Worst case**
Small	500	300	300
Large	100	900	100

A pure maximin decision chooses Small and guarantees 300. But a mixed strategy may guarantee more. Let x be the probability of Small. The guaranteed payoff against Poor is

$$E_P(x) = 500x + 100(1 - x) = 100 + 400x.$$

Against Good,

$$E_G(x) = 300x + 900(1 - x) = 900 - 600x.$$

Equate them:

$$100 + 400x = 900 - 600x \quad \Rightarrow \quad x = 0.8.$$

Worked Example: Worked Example 5: Small or large production

The firm should use Small 80% and Large 20%, guaranteeing

$$V = 100 + 400(0.8) = 420.$$

This is better than the pure maximin guarantee of 300.

Checkpoint: Decision theory or game against nature?

In the production example, suppose reliable probabilities are available: $P(\text{Poor}) = 0.30$ and $P(\text{Good}) = 0.70$. Should the firm still use the conservative mixed strategy? Compute the expected value of Small and Large and compare with the game-theoretic guarantee.

Partial conflict and Nash equilibrium

Core explanation, worked reasoning, and application

Partial conflict and Nash equilibrium

Zero-sum games are only one type of strategic model. In many real situations, both players may benefit from cooperation, but each also has an incentive to defect.
For partial-conflict games, we write payoffs as ordered pairs (row payoff, column payoff).

Concept: Nash equilibrium

A strategy pair is a **Nash equilibrium** if no player can improve their own payoff by switching strategy unilaterally. In a movement diagram, a Nash equilibrium is a cell into which all relevant preference arrows point.

Worked Example: Worked Example 6: Prisoner's Dilemma as an arms race

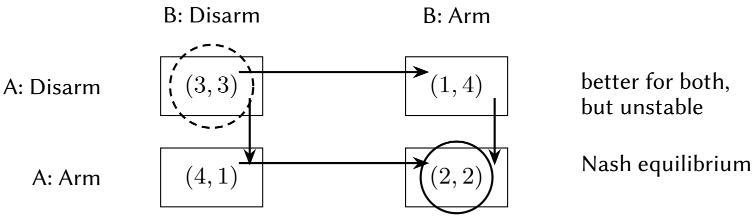
Two countries choose Disarm or Arm. Payoffs are preference scores, where 4 is best and 1 is worst.

Country A \ Country B	**B Disarms**	**B Arms**
A Disarms	$(3,3)$	$(1,4)$
A Arms	$(4,1)$	$(2,2)$

If B disarms, A prefers to arm because $4 > 3$. If B arms, A still prefers to arm because $2 > 1$. Thus Arm is A's dominant strategy. By symmetry, Arm is also B's dominant strategy. The Nash equilibrium is therefore (Arm, Arm) with payoff $(2,2)$.

However, (Disarm, Disarm) gives $(3,3)$, which is better for both. This is the dilemma: individually rational choices lead to a collectively worse outcome.

Illustration: Partial conflict and Nash equilibrium



In the Prisoner's Dilemma, the Nash equilibrium is not the socially best outcome.

Model Critique: A common modeling mistake

Do not call every conflict a Prisoner's Dilemma. A true Prisoner's Dilemma requires: each player has a dominant strategy to defect; mutual defection is the Nash equilibrium; mutual cooperation is better for both than mutual defection. Many conflicts are coordination games, chicken games, or bargaining games instead.

Differentiated strategic modeling studio

Core explanation, worked reasoning, and application

Activity: Strategic modeling studio

Choose one route according to readiness. All groups should finish with a short written recommendation.

Core route: Pricing competition. Two drink stalls choose Low or High prices. The payoffs are weekly profits in hundreds of dollars.

	Stall B Low	Stall B High
Stall A Low	$(6, 6)$	$(10, 3)$
Stall A High	$(3, 10)$	$(8, 8)$

Draw a movement diagram and identify any Nash equilibrium. Is the result socially efficient?

Challenge route: Inspection game. A student may Study or Not Study. A teacher may Check or Not Check. From the student's point of view, payoffs are:

	Check	Not Check
Study	4	3
Not Study	-2	5

Treat this as a zero-sum simplification. Find the student's optimal mixed strategy and the value of the game.

Extension route: General formulation. For the matrix

$$\begin{pmatrix} 5 & 2 & 4 \\ 1 & 6 & 3 \end{pmatrix},$$

Activity: Strategic modeling studio

write the row player's linear program to maximize the guaranteed payoff V. You do not need to solve it, but each constraint must have a clear meaning.

Checkpoint: Competition-style communication

Write a four-sentence conclusion for one game in this lecture:

1. identify the recommended strategy;
2. state the game value or equilibrium outcome;
3. explain the strategic reason behind the recommendation;
4. state one limitation of the model.

Avoid writing only "we choose Row 1" without explaining what the opponent is expected to do.

Lesson summary

Core explanation, worked reasoning, and application

Summary: Eight ideas to retain

1. Game theory models decisions in which other rational players react.
2. A payoff matrix must state whose payoff is being recorded.
3. In a zero-sum game, the row player maximizes and the column player minimizes.
4. A pure-strategy saddle point exists when maximin equals minimax.
5. Without a saddle point, mixed strategies prevent exploitation.
6. In a 2×2 zero-sum game, optimal mixing is found by making the opponent indifferent.
7. In partial-conflict games, Nash equilibria can be stable but inefficient.
8. Game theory gives strategic insight only if the payoff matrix and rationality assumptions are credible.

Checkpoint: Exit ticket

Complete the sentence:
"A game-theory model is useful when ________________________________, but it may fail when ________________________________."

Practice problems

Core explanation, worked reasoning, and application

Exercises: Core practice

1. For each zero-sum matrix, find row minima, column maxima, and decide whether a saddle point exists.

$$A = \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}.$$

2. A goalkeeper dives Left or Right. A kicker shoots Left or Right. Entries are scoring probabilities:

	Goalkeeper L	Goalkeeper R
Kick L	0.55	0.90
Kick R	0.85	0.60

   Check for a saddle point. If none exists, find the kicker's optimal mixed strategy and the game value.

3. A firm faces uncertain demand and chooses Conservative or Aggressive production. Payoffs are in thousands:

	Low demand	High demand
Conservative	40	60
Aggressive	10	100

   Treat demand as nature. Find the best pure maximin decision. Then find whether a mixed strategy can guarantee more.

Exercises: Core practice

4. Draw the movement diagram for the pricing game in the studio. Is there a dominant strategy for either stall? Is the Nash equilibrium Pareto efficient?
5. In the Prisoner's Dilemma matrix from this lecture, explain why (Disarm, Disarm) is not a Nash equilibrium even though it is better for both countries.
6. Give one real-world situation that should *not* be modeled as zero-sum. Explain what would be lost if it were forced into a zero-sum payoff matrix.

Exercises: Modeling challenge

Choose one scenario and prepare a one-page strategic modeling brief containing players, strategies, payoff table, assumptions, equilibrium or recommended strategy, and limitations.

1. Two schools decide whether to host their open day on the same weekend or different weekends.
2. Two delivery companies choose whether to enter a new district.
3. A student team chooses a competition topic while anticipating what other teams may choose.

Your brief should justify the payoff values qualitatively even if exact numerical data are unavailable.

Exercises: Extension practice

1. **Battle of the search routes.** A rescue team searches North or South. A lost hiker chooses, unknowingly, one of two paths. The payoff to the rescue team is probability of finding the hiker within one hour:

	Hiker North	Hiker South
Search North	0.80	0.30
Search South	0.40	0.70

   Treat this as a game against nature. Find the pure maximin strategy and the mixed strategy that maximizes the guaranteed probability of success.
2. **Penalty kick interpretation.** A team claims, “Our kicker should always kick right because the average scoring rate is higher.” Explain why this argument may fail in a strategic game. Use the goalkeeper’s possible response in your explanation.
3. **Dominance with context.** In a pricing game, one row is dominated mathematically. Give a possible real-world reason why a firm might still consider it, and explain whether that means the payoff matrix is incomplete.
4. **Repeated interaction.** Revisit the drink-stall pricing game. Explain how repeated play, reputation, or a school regulation might change the equilibrium outcome. State which assumption of the one-shot game has changed.

Exercises: Strategic modeling brief template

For a competition-style response, students may use the following structure:

1. **Players:** Who makes strategic decisions?
2. **Strategies:** What actions are realistically available to each player?
3. **Payoffs:** What does each player value, and how are numerical payoffs justified?
4. **Solution concept:** Is the game zero-sum, a game against nature, or partial conflict? Which method is appropriate?
5. **Result:** State the saddle point, mixed strategy, game value, or Nash equilibrium.
6. **Interpretation:** What does the result recommend in the real situation?
7. **Limitations:** Which assumptions are most fragile?

Optional extension: Linear programming formulation for larger zero-sum games

Core explanation, worked reasoning, and application

Optional extension: Linear programming formulation for larger zero-sum games

For an $m \times n$ zero-sum matrix $A = (a_{ij})$, let x_i be the probability that the row player chooses row i. The row player maximizes the guaranteed payoff V:

$$\begin{aligned}
\text{maximize} \quad & V \\
\text{subject to} \quad & V \leq \sum_{i=1}^{m} a_{ij} x_i, \qquad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} x_i = 1, \\
& x_i \geq 0.
\end{aligned}$$

Each inequality says: even if the column player chooses column j, the expected payoff must be at least V. The column player's dual problem minimizes V subject to analogous constraints. The equality of the two optimal values is closely related to the minimax theorem.

Optional extension: Repeated Prisoner’s Dilemma

Core explanation, worked reasoning, and application

Optional extension: Repeated Prisoner's Dilemma

If the Prisoner's Dilemma is played once, defection is individually rational. If it is repeated many times, cooperation can become stable. A simple strategy is **Tit-for-Tat**: cooperate first, then copy the opponent's previous move. It rewards cooperation and punishes defection, but only when players expect future interaction and can observe one another's actions reliably.