Lecture 6: Linear Programming and Decision Optimization

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Learning Goals

By the end of the lesson, readers should be able to:

1. identify **decision variables**, an **objective function**, and **constraints** from a real situation;
2. formulate a two-variable **linear program**;
3. graph the **feasible region** and locate its **extreme points**;
4. use corner evaluation and objective level lines to solve both maximization and minimization models;
5. recognise infeasible, unbounded, and multiple-optimum cases from geometry;
6. explain why integrality, linearity, certainty, and divisibility are modeling assumptions rather than facts;
7. formulate a small **binary program** with budget and logical constraints;
8. interpret **binding constraints**, **slack**, **shadow prices**, and a **range of optimality** in context.

From a situation to a decision model

Core explanation, worked reasoning, and application

From a situation to a decision model

Optimization begins with a decision, not an algorithm. A modeler must first decide what can be controlled, what outcome matters, and what limits the possible choices. A general constrained optimization model has the form

$$\text{optimize } f(\mathbf{x}) \qquad \text{subject to} \qquad g_i(\mathbf{x}) \le b_i.$$

When the objective and every constraint are linear, the model is a **linear program**.

Activity: Launch: Planning a production week

A school workshop makes display stands and storage boxes for an exhibition. Each product earns a different contribution, but both use limited wood and labor.

In pairs, decide:

1. what the decision variables should represent;
2. what quantity should be maximized;
3. which resource limits must become constraints;
4. whether fractional answers such as 12.4 boxes are meaningful.

Modeling lesson: The mathematical structure should reflect the decision that must actually be implemented.

Concept: Four ingredients of an optimization model

1. **Decision variables:** quantities that the decision maker controls.
2. **Objective function:** a numerical measure of success to maximize or minimize.
3. **Constraints:** resource, policy, physical, or logical restrictions.
4. **Domain restrictions:** nonnegative, integer, binary, or bounded values required by context.

A model is incomplete if any of these ingredients is left implicit.

Worked Example: Worked Example 1: Formulating the carpenter's problem

A carpenter earns a contribution of \$25 per table and \$30 per bookcase. A table requires 20 board-feet of lumber and 5 labor hours; a bookcase requires 30 board-feet and 4 labor hours. Each week, 690 board-feet and 120 labor hours are available.

Let

$$x = \text{number of tables}, \qquad y = \text{number of bookcases}.$$

The objective is weekly contribution

$$\boxed{\text{maximize } P = 25x + 30y.}$$

The resource constraints are

$$\begin{aligned} 20x + 30y &\leq 690 \quad && \text{(lumber)}, \\ 5x + 4y &\leq 120 \quad && \text{(labor)}, \\ x, y &\geq 0 \quad && \text{(nonnegativity)}. \end{aligned}$$

If only whole products can be made, the additional restrictions $x, y \in \mathbb{Z}$ should be stated. We first solve the continuous model, then examine whether the resulting decision is implementable.

Checkpoint: Formulation

A bakery makes regular cakes and premium cakes. Contributions are \$18 and \$26. Each regular cake requires 2 hours of preparation and 1 hour of decoration; each premium cake requires 3 preparation hours and 2 decoration hours. Weekly capacities are 36 preparation hours and 20 decoration hours.

Define the variables and write the complete linear program. Include units and domain restrictions.

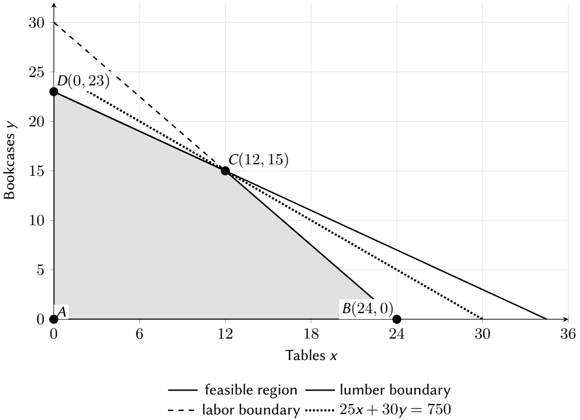
The feasible region

Core explanation, worked reasoning, and application

The feasible region

Each inequality describes a half-plane. The **feasible region** is the set of points satisfying all constraints simultaneously. A feasible point is an implementable plan under the assumptions of the model; an infeasible point violates at least one restriction.

Illustration: The feasible region



The feasible region is the overlap of all constraints. The dotted objective line is the highest parallel level line that still touches the feasible region.

The feasible region

A set is **convex** if the line segment joining any two points in the set remains inside the set. Intersections of linear half-planes are convex, so every linear-programming feasible region is convex.

Concept: Why do corners matter?

A linear objective has parallel level lines. Moving a level line in the improving direction, the last contact with a bounded convex polygon occurs at a corner, or along an entire edge when there are multiple optimal solutions. Therefore, for a two-variable linear program, it is enough to evaluate the objective at the feasible extreme points. This conclusion depends on linearity; it is not generally true for nonlinear objectives.

Worked Example: Worked Example 2: Solving by extreme points

The feasible extreme points are

$$A = (0,0), \quad B = (24,0), \quad C = (12,15), \quad D = (0,23).$$

Evaluate $P = 25x + 30y$:

point	P (dollars)
A	0
B	600
C	750
D	690

Hence the continuous optimum is

$$\boxed{x = 12, \qquad y = 15, \qquad P = 750.}$$

Both resource constraints are equalities at C, so all lumber and all labor are used.

Activity: Guided graphical solution

For the bakery model from the checkpoint:

1. draw each boundary line using intercepts;
2. identify the feasible side of each line;
3. list all feasible extreme points;
4. evaluate the objective at each extreme point;
5. state the optimal plan in a complete sentence.

Minimization and minimum requirements

Core explanation, worked reasoning, and application

Minimization and minimum requirements

Not every optimization problem seeks the largest profit. A school may minimize staffing cost, a transport company may minimize fuel use, or a diet model may minimize cost while meeting minimum nutritional requirements. Constraints of the form “at least” usually produce $\geq$ inequalities, so the feasible region may lie *above* the boundary lines and may not contain the origin.

Worked Example: Worked Example 3: Minimum-cost delivery fleet

A delivery service can hire trucks and vans for one day. A truck costs \$120 and carries 8 pallet units; a van costs \$70 and carries 4 pallet units. At least 48 pallet units must be carried, and at least 8 vehicles must be available for route coverage.
Let x be the number of trucks and y the number of vans. The continuous model is

$$\text{minimize } C = 120x + 70y \quad \text{subject to} \quad 8x + 4y \geq 48, \qquad x + y \geq 8, \qquad x, y \geq 0.$$

Dividing the first constraint by 4 gives $2x + y \geq 12$. The relevant lower-bound extreme points are

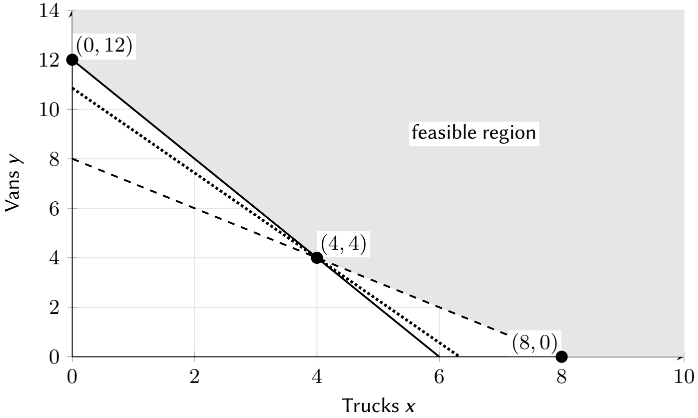
$$(0, 12), \qquad (4, 4), \qquad (8, 0).$$

Their costs are \$840, \$760, and \$960 respectively. Hence the continuous optimum is

$$\boxed{x = 4, \qquad y = 4, \qquad C = 760.}$$

Here the objective line is moved *toward smaller cost* until it first touches the feasible region.

Illustration: Minimization and minimum requirements



For minimum requirements, the feasible region lies above the constraints. The minimum-cost level line first touches the region at $(4, 4)$.

Checkpoint: Maximize or minimize?

For each situation, identify a plausible objective and state whether it should be maximized or minimized:

1. scheduling revision sessions while meeting minimum subject coverage;
2. allocating an advertising budget to increase expected registrations;
3. choosing emergency supplies while keeping total weight as small as possible.

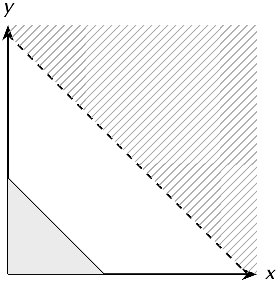
Three special outcomes

Core explanation, worked reasoning, and application

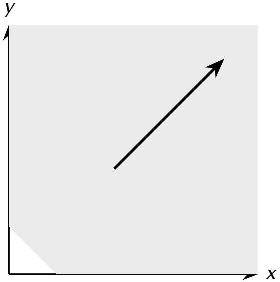
Three special outcomes

A correctly formulated linear program does not always have one finite optimal corner. The geometry should be checked before an answer is reported.

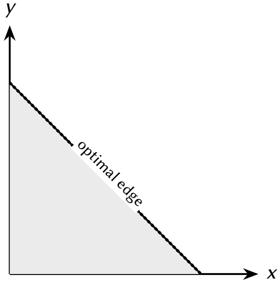
Illustration: Three special outcomes



Infeasible



Unbounded



Multiple optima

required regions do not overlap

objective can improve indefinitely

objective is parallel to an edge

Before interpreting a solution, check whether the model is infeasible, unbounded, or has an entire edge of optimal solutions.

Concept: How to report special cases

- **Infeasible:** the requirements cannot all be satisfied. Recheck data or negotiate constraints.
- **Unbounded:** the model permits indefinite improvement. A missing upper limit, cost, or capacity is often the cause.
- **Multiple optima:** several decisions are mathematically equivalent, so secondary criteria such as risk, fairness, or ease of implementation can decide among them.

Model Critique: A feasible point is not automatically a sensible decision

The mathematics checks only the stated constraints. A feasible production plan may still be unacceptable because the model omitted demand limits, storage, contractual minimums, setup costs, uncertainty, or fairness. Optimization does not replace judgment; it makes the stated trade-offs explicit.

Binding constraints, slack, and what-if questions

Core explanation, worked reasoning, and application

Binding constraints, slack, and what-if questions

At a proposed solution, a constraint is **binding** if it holds with equality. Otherwise it has **slack**. For a resource constraint

$$a_1 x + a_2 y \leq b,$$

the slack is $b - (a_1 x + a_2 y)$.

Worked Example: Worked Example 4: Interpreting slack

At $B = (24, 0)$:

$$\text{lumber used} = 20(24) = 480, \quad \text{lumber slack} = 690 - 480 = 210,$$
$$\text{labor used} = 5(24) = 120, \qquad \text{labor slack} = 120 - 120 = 0.$$

Labor is binding, while 210 board-feet remain unused. Adding lumber alone cannot improve this plan locally because labor is already exhausted.

At the optimum $C = (12, 15)$, both slacks are zero. Both resources are bottlenecks under the current coefficients.

Binding constraints, slack, and what-if questions

A **shadow price** estimates the increase in the optimal objective value from one additional unit of a binding resource, provided the change is small enough that the same pair of constraints remains active.

Worked Example: Worked Example 5: Marginal value of resources

At the carpenter optimum, the two binding constraints are

$$20x + 30y = b_L, \qquad 5x + 4y = b_H.$$

The shadow prices u and v for lumber and labor satisfy

$$20u + 5v = 25, \qquad 30u + 4v = 30,$$

because the value assigned to the resources used by one product must match its contribution at the margin. Solving gives

$$\boxed{u = \frac{5}{7} \approx 0.71 \text{ dollars per board-foot}}, \qquad \boxed{v = \frac{15}{7} \approx 2.14 \text{ dollars per labor hour}}.$$

Thus one extra labor hour is locally more valuable than one extra board-foot. Paying \$3 for one extra labor hour would not be worthwhile under this local model, while paying \$1 might be.

Worked Example: Worked Example 6: Range of optimality for table profit

Suppose the table contribution changes from \$25 to c_T, while bookcase contribution remains \$30. The objective level lines have slope

$$-\frac{c_T}{30}.$$

At the current optimum $C = (12, 15)$, the slopes of the two binding edges are $-5/4$ (labor) and $-2/3$ (lumber). The same corner remains optimal while the objective slope lies between them:

$$-\frac{5}{4} \leq -\frac{c_T}{30} \leq -\frac{2}{3}.$$

Reversing signs and solving gives

$$\boxed{20 \leq c_T \leq 37.5.}$$

Within this interval, the recommended production quantities remain $(12, 15)$ even though total contribution changes. At an endpoint, the objective line is parallel to one edge and multiple optimal solutions appear. Outside the interval, a different corner becomes optimal.

Activity: Sensitivity studio

Use slopes rather than recomputing every objective value.

1. If table contribution rises to \$34, does the production mix change?
2. If it rises to \$40, which neighboring corner should become preferable?
3. Why is a **range of optimality** more useful than testing only a single alternative coefficient?

Checkpoint: Slack and marginal value

A solution leaves 40 kg of material unused but uses every available labor hour.

1. Which constraint is binding?
2. Which resource should have zero shadow price locally?
3. Why does “buy the resource with the larger shadow price” still require a validity range and a cost comparison?

Model Critique: Sensitivity is local

A shadow price is not a universal market value. If a resource changes enough, the optimal corner may change and the old shadow price no longer applies. Sensitivity analysis should report both the marginal value and the range over which the interpretation remains valid.

When whole-number decisions matter

Core explanation, worked reasoning, and application

When whole-number decisions matter

A continuous linear program allows fractional values. This is appropriate for divisible quantities such as kilograms, hours, or proportions. It may be inappropriate for people, vehicles, projects, or indivisible products. Adding $x_j \in \mathbb{Z}$ gives an **integer program**; restricting $x_j \in \{0, 1\}$ gives a **binary program**.

Worked Example: Worked Example 7: Why rounding can fail

Consider

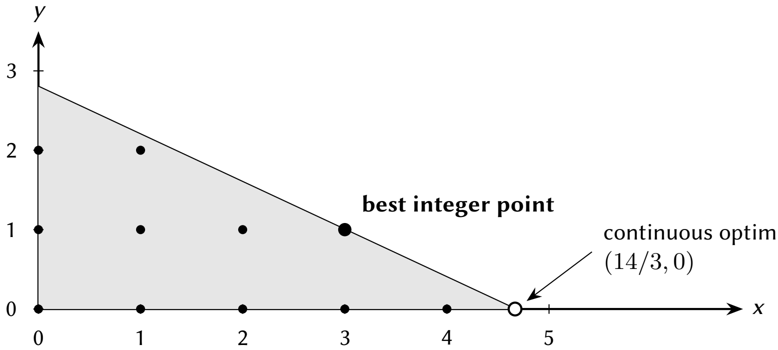
$$\text{maximize } 7x + 10y \quad \text{subject to} \quad 3x + 5y \leq 14, \qquad x, y \geq 0,$$

where x and y count indivisible packages.

The continuous model prefers $(14/3, 0)$, with objective $98/3 \approx 32.67$. Rounding x to 5 violates the capacity constraint because $3(5) = 15 > 14$. Rounding down to $(4, 0)$ gives value 28, but the integer-feasible point $(3, 1)$ uses all capacity and gives value 31.

Conclusion: Solving continuously and rounding afterward can produce either an infeasible or a suboptimal decision.

Illustration: When whole-number decisions matter



Integer feasibility replaces a continuous region by a set of lattice points. The best integer point need not be obtained by rounding the continuous optimum.

Worked Example: Worked Example 8: Binary project selection

A student council considers five projects. Costs and estimated benefit scores are:

Project	Cost	Benefit score
A: Study hub	4	8
B: Booking system	5	12
C: Peer mentoring	3	7
D: Wellbeing event	4	9
E: Equipment fund	6	11

Let $z_A, \dots, z_E \in \{0, 1\}$ indicate selection. The council has budget 12, may choose at most three projects, Project B requires A, and C and D cannot both be chosen:

$$\text{maximize } B = 8z_A + 12z_B + 7z_C + 9z_D + 11z_E,$$

$$\begin{aligned} 4z_A + 5z_B + 3z_C + 4z_D + 6z_E &\le 12, \\ z_A + z_B + z_C + z_D + z_E &\le 3, \\ z_B &\le z_A, \\ z_C + z_D &\le 1. \end{aligned}$$

Worked Example: Worked Example 8: Binary project selection

For this small problem, feasible combinations can be enumerated. The strongest candidates are

selection	cost	benefit
$A+B+C$	12	27
$D+E$	10	20
$A+E$	10	19
$C+E$	9	18
$A+D$	8	17

Hence the optimal selection is $A+B+C$, with benefit score 27.

Activity: Differentiated optimization studio

Choose one route.
Core route: Verify every constraint for the proposed choice $A + B + C$, then explain why choosing B alone is prohibited.
Challenge route: Suppose the budget falls to 10. Find the new optimal combination by systematic enumeration and report which logical constraint becomes important.
Extension route: Add a condition "exactly one of D and E must be selected" and write the corresponding binary equation. Discuss whether the original optimum remains feasible.

Assumptions and communication

Core explanation, worked reasoning, and application

Assumptions and communication

The standard linear-programming assumptions are useful but demanding:

- **Proportionality:** doubling an activity doubles its resource use and contribution.
- **Additivity:** total effects are sums; there are no interaction or setup effects.
- **Divisibility:** continuous variables may take fractional values.
- **Certainty:** all coefficients are treated as known constants.

Model Critique: When linearity becomes questionable

Bulk discounts, overtime premiums, fixed setup costs, uncertain demand, product interactions, congestion, and economies of scale all violate simple linearity. A useful first model may still be linear, but its assumptions and intended range must be stated explicitly.

Activity: Competition-style conclusion

Write a four-sentence recommendation for the carpenter:

1. report the optimal production plan and contribution;
2. identify the binding resources;
3. state one useful sensitivity result;
4. acknowledge one assumption that could change the decision.

Avoid writing that the model “proves” the decision is best in reality.

Lesson summary

Core explanation, worked reasoning, and application

Summary: Nine ideas to retain

1. Optimization translates a decision into variables, an objective, constraints, and domain restrictions.
2. A feasible solution satisfies every stated constraint; feasibility does not guarantee practical suitability.
3. A two-variable linear program can be solved by evaluating feasible extreme points, whether the objective is maximized or minimized.
4. A model may be infeasible, unbounded, or have multiple optimal solutions; each outcome has a different interpretation.
5. Binding constraints identify current bottlenecks; slack measures unused capacity.
6. Shadow prices provide local marginal values, not permanent prices.
7. A range of optimality describes how far an objective coefficient can change before the recommended corner changes.
8. Whole-number or yes/no decisions require integer or binary variables; naive rounding is unsafe.
9. Logical requirements such as dependence and incompatibility can be written as linear constraints on binary variables.
10. The optimal answer is conditional on assumptions about linearity, divisibility, certainty, and omitted factors.

Checkpoint: Exit ticket

Complete both statements:
“The mathematical optimum is ______________________________.”
“Before implementing it, I would check ____________________________________.”

Practice problems

Core explanation, worked reasoning, and application

Exercises: Core practice

1. A farmer plants wheat and corn. Wheat earns \$200 per acre and uses 3 labor-days and 2 fertilizer units. Corn earns \$300 per acre and uses 2 labor-days and 4 fertilizer units. At most 45 acres, 100 labor-days, and 120 fertilizer units are available.
   1.1 Define the variables and formulate the linear program.
   1.2 Graph the feasible region and find all extreme points.
   1.3 Determine the optimal planting plan and identify binding constraints.
2. For the carpenter model, calculate the slack in each resource at A, B, C, and D. Explain why unused lumber at B does not imply that B can be improved by adding more tables.
3. A delivery plan uses trucks x and vans y:

$$\text{minimize } 120x + 70y \quad \text{subject to} \quad 8x + 4y \geq 48, \quad x + y \geq 8, \quad x, y \geq 0.$$

   Interpret every coefficient, graph the feasible region, and find the continuous optimum. Then discuss whether integrality matters.
4. A student claims: "The point with the greatest x-coordinate must maximize every profit function." Explain the error using level lines.

Exercises: Core practice

5. For each of the following, sketch a feasible region and explain the outcome: (a) inconsistent constraints, (b) an unbounded maximization problem, and (c) multiple optimal solutions.
6. For the carpenter model, derive the range of bookcase contribution c_B for which $(12, 15)$ remains optimal. Interpret what happens at the endpoints.
7. Re-solve the binary project model when the budget is 10. Show a compact enumeration table rather than listing all 2^5 possibilities.
8. Construct an example in which a linear program has multiple optimal solutions. State the geometric condition that causes this.
9. A report states: "One extra labor hour is worth \$2.14, so 1000 additional hours are worth \$2140." Critique the statement carefully.

Exercises: Modeling challenge

A school must choose how many standard and intensive revision workshops to offer. Each type uses teacher hours, rooms, and a limited printing budget. Student benefit is estimated differently for the two formats. Prepare a one-page optimization brief containing:

1. decision purpose and stakeholders;
2. variables with units and domain restrictions;
3. an objective and at least three constraints;
4. assumptions behind linearity and benefit scores;
5. a graphical or computational solution plan;
6. one sensitivity question and one implementation limitation.

The quality of the formulation and interpretation is more important than obtaining a particular numerical optimum.

Optional extension: What the Simplex Method does

Core explanation, worked reasoning, and application

Optional extension: What the Simplex Method does

The **Simplex Method** is an algebraic method for larger linear programs. It starts at one feasible extreme point and repeatedly moves to an adjacent extreme point with a better objective value. **Slack variables** convert inequalities into equations, and a **pivot** changes the current basis.
For the carpenter model, Simplex follows the path

$$(0,0) \longrightarrow (0,23) \longrightarrow (12,15),$$

then stops because no adjacent move improves the objective. A full tableau treatment is best taught as a separate lesson after students understand formulation, geometry, special cases, and interpretation.

Optional extension: Derivative-free search

Core explanation, worked reasoning, and application

Optional extension: Derivative-free search

A one-dimensional nonlinear objective may be optimized by **golden-section search** when it is unimodal on a known interval. This is a different topic from linear programming: it narrows an interval containing one optimum rather than moving among corners of a feasible polytope. It is reserved for a later numerical-optimization lesson.