Lecture 4: Empirical Models, Interpolation, and Splines

Mathematical Modeling Lecture

Mathematical Modeling

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Session Route

Launch

Question

Motivating context, key vocabulary, and the first modelling decision.

Build

Method

Worked example, diagram or table, calculation, and parameter meaning.

Test

Critique

Checkpoint practice, limitation check, and communication of results.

Learning Goals

By the end of the lesson, readers should be able to:

1. distinguish **interpolation**, **smoothing**, and **extrapolation**;
2. construct and interpret a linear interpolant between two observations;
3. build a quadratic interpolating polynomial in Lagrange form;
4. use a difference table to recognise simple polynomial structure;
5. explain why an exact high-degree interpolant may behave poorly between data points;
6. compare piecewise linear interpolation with a natural cubic spline and select an appropriate empirical method.

When the data come before the formula

Core explanation, worked reasoning, and application

When the data come before the formula

Lecture 3 began with a proposed model family and estimated its parameters from data. In many practical situations, however, no convincing mechanism is yet available. We may only have measurements at selected times or locations and need a reasonable value between them. This leads to an **empirical model**: a mathematical representation constructed mainly from observed data.

Activity: Launch: Should the curve pass through every point?

A classroom sensor records carbon-dioxide concentration once every five minutes:

time (min)	0	5	10	15	20
CO_2 (ppm)	612	648	701	694	760

In pairs, discuss:

1. How would you estimate the concentration at minute 12?
2. Should the estimated curve pass through every recorded value?
3. Which matters more here: measurement noise or exact agreement with the observations?
4. Would the same method be defensible for predicting minute 60?

Modeling lesson: The purpose of the model and the reliability of the data determine whether exact interpolation is desirable.

Concept: Interpolation, smoothing, and extrapolation

- **Interpolation** constructs a function that agrees with specified data values and estimates values *inside* the observed range.
- **Smoothing** allows small deviations from noisy data to reveal an underlying trend.
- **Extrapolation** predicts *outside* the observed range. It depends on assumptions that the data alone cannot verify.

An interpolant can be mathematically exact and still be scientifically untrustworthy if the observations contain noise or the curve behaves implausibly between them.

Model Critique: Exact agreement is not the same as credibility

If a thermometer is accurate only to the nearest degree, forcing a complicated curve through every reported value may model rounding noise rather than temperature. Conversely, if the data are exact geometric coordinates in computer graphics, passing through every point may be essential. The data-generating process must guide the method.

Linear interpolation and piecewise linear models

Core explanation, worked reasoning, and application

Linear interpolation and piecewise linear models

Suppose two trusted observations are (x_0, y_0) and (x_1, y_1), with $x_0 < x_1$. For a point x between them, define the relative position

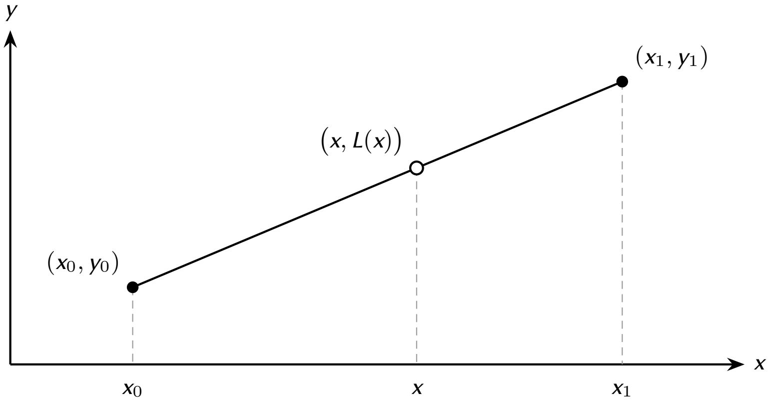
$$\lambda = \frac{x - x_0}{x_1 - x_0}, \qquad 0 \le \lambda \le 1.$$

The **linear interpolant** is

$$\boxed{L(x) = (1 - \lambda)y_0 + \lambda y_1 = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).}$$

The formula is a weighted average: when x is closer to x_0, more weight is placed on y_0.

Illustration: Linear interpolation and piecewise linear models



The interpolated point lies on the line segment joining the two observations; its horizontal position determines the weights assigned to y_0 and y_1.

Worked Example: Worked Example 1: Estimating temperature between readings

A heating experiment records $38.0°\mathrm{C}$ at $t = 4$ minutes and $62.0°\mathrm{C}$ at $t = 10$ minutes. Estimate the temperature at $t = 7.5$ minutes.
The relative position is

$$\lambda = \frac{7.5 - 4}{10 - 4} = \frac{3.5}{6}.$$

Hence

$$T(7.5) = 38.0 + \frac{3.5}{6}(62.0 - 38.0) = 38.0 + 14.0 = \boxed{52.0°\mathrm{C}}.$$

The estimate assumes that temperature changed at an approximately constant rate over this six-minute interval. It does not claim that the entire heating process is linear.

Linear interpolation and piecewise linear models

With several observations, we may connect each consecutive pair by a line. The resulting **piecewise linear interpolant** is continuous and simple to compute, but its slope changes abruptly at the data points.

Activity: Guided practice: Build a piecewise model

A library records the number of occupied seats:

$$(0, 20), \qquad (2, 44), \qquad (5, 68),$$

where x is hours after opening.

1. Write the linear formula on $0 \leq x \leq 2$.
2. Write the linear formula on $2 \leq x \leq 5$.
3. Estimate occupancy at $x = 1.5$ and $x = 3.5$.
4. Compare the slopes of the two pieces. What does the change of slope mean in context?

Checkpoint: Interpolation or extrapolation?

For the temperature observations at $t = 4$ and $t = 10$ minutes, classify each request:

1. estimate $T(8)$;
2. estimate $T(15)$;
3. estimate the maximum temperature after one hour.

Explain why only one request can be answered by linear interpolation alone.

Quadratic interpolation through three points

Core explanation, worked reasoning, and application

Quadratic interpolation through three points

Two points determine a line; three points with distinct x-coordinates determine a unique polynomial of degree at most two. The **Lagrange form** constructs this polynomial directly.
For three points (x_0, y_0), (x_1, y_1), and (x_2, y_2),

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x),$$

where

$$\begin{aligned} L_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \\ L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}, \\ L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned}$$

Each basis polynomial satisfies $L_i(x_i) = 1$ and is zero at the other two nodes.

Worked Example: Worked Example 2: A quadratic interpolant

Construct the polynomial through

$$(0,1), \qquad (1,3), \qquad (2,2).$$

The basis functions are

$$L_0(x) = \frac{(x-1)(x-2)}{2}, \qquad L_1(x) = -x(x-2), \qquad L_2(x) = \frac{x(x-1)}{2}.$$

Therefore

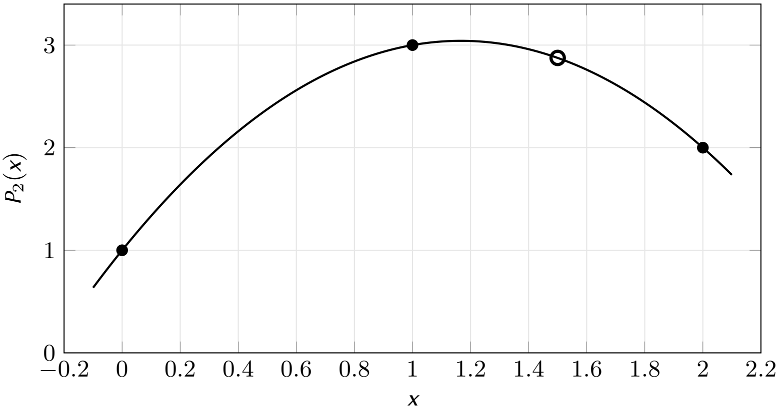
$$\begin{aligned} P_2(x) &= L_0(x) + 3L_1(x) + 2L_2(x) \\ &= -\frac{3}{2}x^2 + \frac{7}{2}x + 1. \end{aligned}$$

At $x = 1.5$,

$$P_2(1.5) = \boxed{2.875}.$$

Substitution verifies $P_2(0) = 1$, $P_2(1) = 3$, and $P_2(2) = 2$ exactly.

Illustration: Quadratic interpolation through three points



The solid curve is $P_2(x) = -1.5x^2 + 3.5x + 1$. Filled points are the interpolation nodes; the open point is the estimate at $x = 1.5$.

Model Critique: What does the curve between the points mean?

The interpolation conditions determine the quadratic uniquely, but they do not prove that the real process is quadratic. A different physical mechanism could produce the same three measurements and different values between them. Interpolation supplies a convenient curve; scientific interpretation still requires context.

Difference tables as a diagnostic

Core explanation, worked reasoning, and application

Difference tables as a diagnostic

For equally spaced values of x, successive differences can reveal low-degree polynomial structure:

- constant first differences suggest a linear model;
- constant second differences suggest a quadratic model;
- constant third differences suggest a cubic model.

The result is exact for perfect polynomial data and only approximate for noisy observations.

Worked Example: Worked Example 3: Recognising a quadratic pattern

Consider

x	0	1	2	3	4
y	2	5	10	17	26

The difference table is

y	2	5	10	17	26
Δy		3	5	7	9
$\Delta^2 y$			2	2	2

The second differences are constant, so the data lie on a quadratic. For unit spacing, the leading coefficient is half the constant second difference, giving $a = 1$. Writing

$$y = x^2 + bx + c$$

and using $y(0) = 2$ and $y(1) = 5$ gives $c = 2$ and $b = 2$. Thus

$$\boxed{y = x^2 + 2x + 2.}$$

Concept: What a difference table can and cannot do

A difference table is a diagnostic, not a certificate of truth. With measured data, higher-order differences often amplify noise. The method is most useful for identifying a simple pattern in accurate, equally spaced data. For unequally spaced nodes, use **divided differences**; an optional introduction appears in Appendix 10.

Activity: Pattern or noise?

8 min

For each sequence, calculate enough differences to suggest a simple model.

1. $4,\ 9,\ 16,\ 25,\ 36$
2. $3.0,\ 6.1,\ 11.9,\ 21.2,\ 34.0$

For the second sequence, explain why “approximately quadratic” is more defensible than “exactly quadratic.”

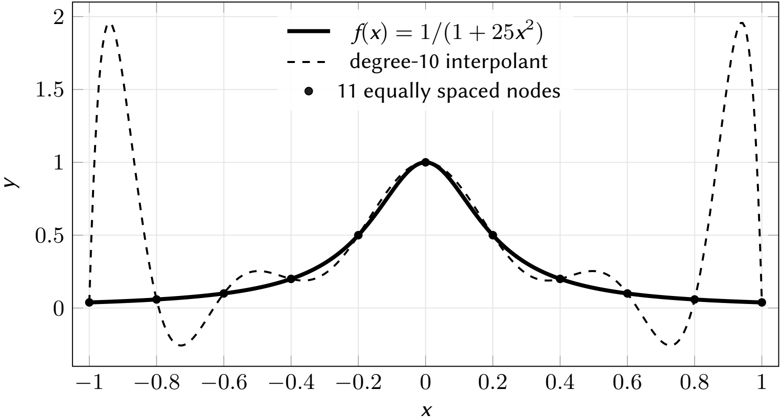
Why a high-degree polynomial can fail

Core explanation, worked reasoning, and application

Why a high-degree polynomial can fail

Given $n+1$ distinct points, a unique polynomial of degree at most n passes through all of them. This theorem guarantees existence, not good behavior. As the degree increases, a global polynomial can oscillate strongly between nodes, especially near the endpoints.

Illustration: Why a high-degree polynomial can fail



The polynomial matches every node but oscillates badly between them. This is **Runge's phenomenon**.

Model Critique: The zero-residual trap

At the interpolation nodes, every residual is zero. Therefore SSE, MAE, and RMSE computed only at those nodes are also zero, even though the curve is poor between them. Model assessment must consider behavior between observations, sensitivity to data perturbations, and the intended use of the curve.

Why a high-degree polynomial can fail

Possible remedies include using fewer degrees of freedom, smoothing noisy data, choosing better-distributed nodes, or replacing one global polynomial with piecewise low-degree polynomials.

Piecewise interpolation and cubic splines

Core explanation, worked reasoning, and application

Piecewise interpolation and cubic splines

A **spline** uses a separate polynomial on each interval between consecutive **knots**. A linear spline joins the points by straight segments. A **cubic spline** uses cubic pieces while imposing smooth joins.
For a cubic spline S:

1. S passes through every data point;
2. adjacent pieces have the same value at each interior knot;
3. their first derivatives agree, so there is no corner;
4. their second derivatives agree, so curvature changes smoothly.

A **natural cubic spline** additionally satisfies $S'' = 0$ at the two endpoints.

Worked Example: Worked Example 4: Natural spline through three points

Construct the natural cubic spline through

$$(0,0), \qquad (1,1), \qquad (2,0).$$

Let $M_i = S''(x_i)$. Natural endpoint conditions give $M_0 = M_2 = 0$. With equal spacing $h = 1$, the single interior spline equation is

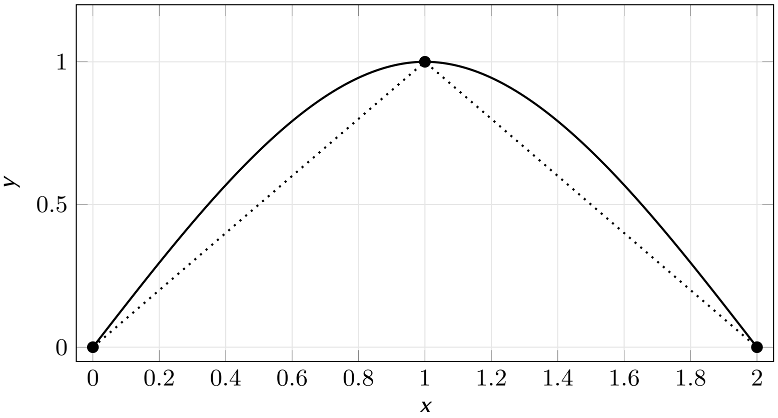
$$M_0 + 4M_1 + M_2 = 6(y_2 - 2y_1 + y_0) = -12,$$

so $M_1 = -3$. The resulting spline is

$$S(x) = \begin{cases} -\frac{1}{2}x^3 + \frac{3}{2}x, & 0 \le x \le 1, \\ 1 - \frac{3}{2}(x-1)^2 + \frac{1}{2}(x-1)^3, & 1 \le x \le 2. \end{cases}$$

At $x = 0.5$, the spline gives $S(0.5) = 0.6875$, while linear interpolation gives 0.5. Neither value is automatically "correct"; the smoother cubic estimate encodes stronger assumptions about how the slope changes.

Illustration: Piecewise interpolation and cubic splines



The dotted line is the linear spline, the solid curve is the natural cubic spline, and the filled points are the knots. Both interpolate the same data, but only the cubic spline has continuous slope and curvature at $x = 1$.

Choosing an empirical method

Core explanation, worked reasoning, and application

Choosing an empirical method

Method	Strength	Main caution
Linear interpolation	Transparent, local, easy to compute	Corners at knots; assumes constant rate within each interval
Quadratic or low-degree interpolation	Captures curvature with few points	A small dataset does not prove the polynomial mechanism
High-degree global polynomial	Passes through every point	Oscillation, sensitivity, and poor extrapolation
Low-degree smoothing model	Handles noisy data and reveals trend	Does not reproduce every observation exactly
Cubic spline	Local, smooth, accurate between many trusted points	May overshoot; boundary conditions matter

Checkpoint: Which property matters?

A road profile is measured at survey points and will be used to estimate vehicle motion. Would you prefer a linear spline or a cubic spline? State one benefit and one risk of your choice.

Model Critique: Cubic splines are not automatically shape-preserving

A cubic spline is smooth and usually stable, but it may overshoot between data points or create negative values when the modeled quantity must remain nonnegative. For monotone or bounded data, specialised shape-preserving splines may be more appropriate.

Activity: Method-selection studio

Choose an appropriate method for each situation and justify it in one or two sentences.

1. Exact coordinates define the edge of a manufactured part.
2. Daily demand data contain substantial random variation, and the goal is to reveal the weekly trend.
3. A temperature sensor failed for two minutes between two reliable readings.
4. Ten trusted measurements describe a smooth road elevation profile.
5. A team wants to forecast ten years beyond a five-year dataset.

For the final situation, explain why none of the interpolation methods is sufficient by itself.

Concept: A practical empirical-modeling workflow

1. Clarify whether the task is interpolation, smoothing, or extrapolation.
2. Plot the original data and check units, ordering, missing values, and outliers.
3. Decide whether observations should be treated as exact or noisy.
4. Start with the simplest local method that meets the purpose.
5. Inspect the curve between points, not only at the observations.
6. Test sensitivity to removing or perturbing a data point.
7. State the valid interval and any shape constraints explicitly.

Lesson summary

Core explanation, worked reasoning, and application

Summary: Six ideas to retain

1. Interpolation estimates inside the data range; extrapolation requires additional assumptions.
2. Linear interpolation is a weighted average between two observations.
3. Three distinct nodes determine a unique quadratic, but not necessarily the true mechanism.
4. Difference tables can reveal simple polynomial patterns in equally spaced data.
5. Zero residuals at interpolation nodes do not guarantee good behavior between them.
6. Splines replace one unstable high-degree polynomial with smooth, local, low-degree pieces.

Checkpoint: Exit ticket

Complete both statements:

1. "I would interpolate rather than smooth when ____."
2. "A curve passing through every point can still be poor because ____."

Practice problems

Core explanation, worked reasoning, and application

Exercises: Core practice

1. A sensor reads 18.2 units at $t = 3$ and 25.4 units at $t = 9$.
   1.1 Use linear interpolation to estimate the value at $t = 5.5$.
   1.2 State the assumption behind the calculation.
   1.3 Explain why the same formula should not automatically be used at $t = 15$.
2. Construct the piecewise linear interpolant through $(0, 2)$, $(2, 6)$, and $(5, 3)$. Evaluate it at $x = 1$, $x = 3$, and $x = 4.5$. Identify the point where the slope changes.
3. Find the quadratic interpolating polynomial through $(0, 2)$, $(1, 1)$, and $(3, 5)$ using the Lagrange form. Evaluate the polynomial at $x = 2$ and verify all three interpolation conditions.
4. Construct a difference table for
$$7,\ 12,\ 19,\ 28,\ 39,\ 52.$$
   Suggest a polynomial degree and find a formula in terms of $n = 0, 1, 2, \ldots$.
5. A degree-12 polynomial passes through 13 experimental measurements exactly. Give three reasons why a lower-degree smoother or spline may nevertheless be more credible.
6. For the points $(0, 0)$, $(1, 2)$, and $(2, 0)$:
   6.1 write the linear spline;

Exercises: Core practice

6.2 describe the continuity properties required of a cubic spline;
6.3 without calculating the full cubic spline, predict how its graph would differ near $x = 1$.

7. Decide whether interpolation, smoothing, or neither is the main task:

   7.1 estimating rainfall at an unmonitored hour between two observations;
   7.2 finding the long-run trend in noisy monthly sales;
   7.3 predicting population in 2070 using records ending in 2025;
   7.4 reconstructing a computer-drawn curve from exact control points.

Exercises: Modeling challenge

A school records pedestrian flow at a corridor entrance every five minutes, but several readings are missing.
Some recorded values may also contain counting errors.
Prepare a one-page empirical-modeling brief containing:

1. the purpose of reconstructing the missing data;
2. which observations you would treat as reliable or noisy;
3. a comparison of piecewise linear interpolation, cubic splines, and smoothing;
4. one diagnostic graph or calculation;
5. the interval on which your result is valid;
6. one sensitivity test and one limitation.

The quality of the justification matters more than using the most complicated method.

Optional extension: Newton divided differences

Core explanation, worked reasoning, and application

Optional extension: Newton divided differences

For unequally spaced nodes, ordinary differences are replaced by **divided differences**:

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},$$

$$f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

The Newton interpolating polynomial is

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n] \prod_{j=0}^{n-1} (x - x_j).$$

Unlike the Lagrange form, the Newton form can be extended efficiently when a new data point is added.

Worked Example: Unequally spaced data

For $(0,1)$, $(1,3)$, and $(3,7)$,

$$f[0,1]=2, \qquad f[1,3]=2,$$

so

$$f[0,1,3]=\frac{2-2}{3-0}=0.$$

Hence $P_2(x)=1+2x$, showing that the three points are actually collinear even though the spacing is unequal.

Optional extension: General natural-spline system

Core explanation, worked reasoning, and application

Optional extension: General natural-spline system

Let $x_0 < \cdots < x_n$, $h_i = x_{i+1} - x_i$, and $M_i = S''(x_i)$. For a natural cubic spline, $M_0 = M_n = 0$, and the interior second derivatives satisfy

$$h_{i-1}M_{i-1} + 2(h_{i-1} + h_i)M_i + h_iM_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right).$$

This is a tridiagonal linear system, so a spline with many knots can be computed efficiently. The formula is useful for implementation but is not part of the two-hour core lesson.