

Lectures in Mathematical Modeling

Supplementary 7: Definite Integrals, Accumulation, and Numerical Integration

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Learning goals

By the end of this two-hour supplementary lesson, students should be able to:

- interpret a **definite integral** as signed area, accumulated change, and average value;
- construct and read a **Riemann sum** using left, right, and midpoint rectangles;
- use the **Fundamental Theorem of Calculus** to evaluate basic definite integrals;
- apply **substitution** and **integration by parts** in simple modeling cases;
- recognise when an integral is **improper** and whether it converges;
- compare rectangle, trapezoidal, Simpson, and Monte Carlo thinking as numerical integration tools.

1 From area to accumulated change

A definite integral is not just a way to compute geometric area. In modeling it often represents an accumulated quantity: total water entering a tank, total cost over time, average inventory, probability mass, or total pollutant load. The same notation

$$\int_a^b f(x) dx$$

may mean area, total change, or average effect depending on what f represents.

Launch activity: what does the integral measure?

Suppose $r(t)$ is the arrival rate of students at a cafeteria, measured in students per minute. What does

$$\int_0^{20} r(t) dt$$

represent? It is not a rate. It is the total number of students who arrive during the first 20 minutes.

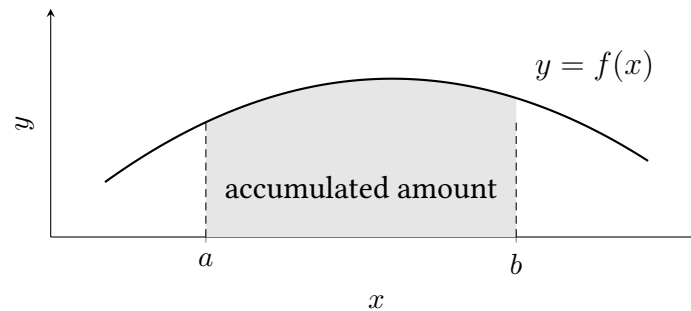


Figure 1: The definite integral accumulates the vertical quantity $f(x)$ across an interval. If f is a rate, the integral is total change. If f is a density, the integral is total mass.

2 Riemann sums: building the integral

Partition and Riemann sum

A **partition** of $[a, b]$ is a list

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The width of the k -th subinterval is $\Delta x_k = x_k - x_{k-1}$. If x_k^* is a sample point in the subinterval, the corresponding **Riemann sum** is

$$\sum_{k=1}^n f(x_k^*) \Delta x_k.$$

For a regular partition, $\Delta x = (b - a)/n$. Common choices are left endpoint, right endpoint, and midpoint.

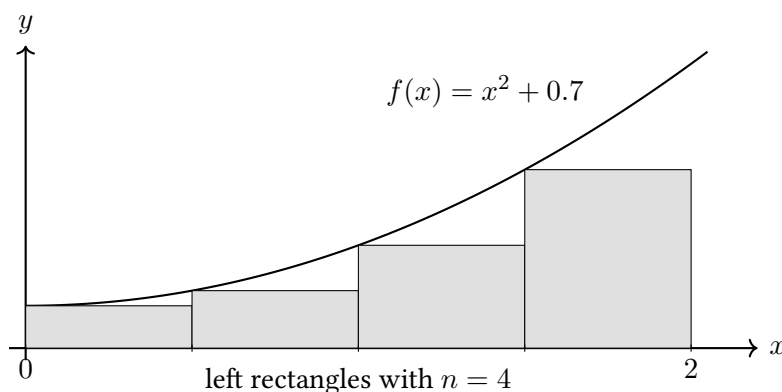


Figure 2: A left Riemann sum uses the height at the left endpoint of each subinterval. For an increasing function, it underestimates the area.

Worked example 1: a left Riemann sum

Approximate

$$\int_0^2 (x^2 + 1) dx$$

using a left Riemann sum with $n = 4$ equal subintervals.

Here $\Delta x = 0.5$, and the left sample points are 0, 0.5, 1, 1.5. Thus

$$S_L = 0.5 [f(0) + f(0.5) + f(1) + f(1.5)] = 0.5(1 + 1.25 + 2 + 3.25) = 3.75.$$

The exact value will later be $14/3 \approx 4.667$, so the left sum underestimates.

The definite integral

If the Riemann sums approach a unique number as the mesh of the partition tends to zero, we define

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k.$$

When $f \geq 0$, this is area. When f changes sign, it is **signed area**: area above the x -axis contributes positively, and area below contributes negatively.

Checkpoint: Checkpoint

For an increasing positive function on $[a, b]$, which is usually larger: the left sum or the right sum? Explain using rectangles, not formulas.

3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the two ideas from Supplementary 5 and this lesson: derivative and integral.

Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$.

• **Part I:** If $A(x) = \int_a^x f(t) dt$, then $A'(x) = f(x)$.

• **Part II:** If $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The first part says accumulation has rate $f(x)$; the second part says definite integrals can be evaluated by antiderivatives.

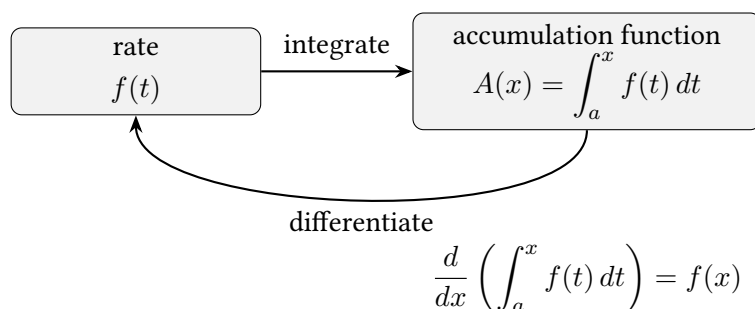


Figure 3: FTC Part I says that integration and differentiation undo each other in the accumulation-function setting.

Worked example 2: evaluating by the FTC

Evaluate

$$\int_0^2 (x^2 + 1) dx.$$

An antiderivative is $F(x) = x^3/3 + x$. Hence

$$\int_0^2 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_0^2 = \frac{8}{3} + 2 = \frac{14}{3}.$$

This explains why the $n = 4$ left sum 3.75 was only an approximation.

Worked example 3: differentiating an integral

Compute

$$\frac{d}{dx} \left(\int_2^{x^2} \sin(t^2) dt \right).$$

Let $u = x^2$. By FTC Part I and the chain rule,

$$\frac{d}{dx} \int_2^u \sin(t^2) dt = \sin(u^2) \frac{du}{dx} = 2x \sin(x^4).$$

The integrand need not have an elementary antiderivative; the derivative is still easy.

4 Substitution and integration by parts

Substitution as reverse chain rule

If $u = g(x)$ and $du = g'(x) dx$, then

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

For a definite integral, change the limits: $x = a \mapsto u = g(a)$ and $x = b \mapsto u = g(b)$.

Worked example 4: substitution with definite limits

Evaluate

$$\int_0^2 x e^{x^2} dx.$$

Let $u = x^2$, so $du = 2x dx$ and $x dx = \frac{1}{2} du$. The limits become $0 \mapsto 0$ and $2 \mapsto 4$. Therefore

$$\int_0^2 x e^{x^2} dx = \frac{1}{2} \int_0^4 e^u du = \frac{1}{2}(e^4 - 1).$$

Integration by parts as reverse product rule

From $(uv)' = u'v + uv'$, we obtain

$$\int u dv = uv - \int v du.$$

A useful choice rule is LIATE: logarithmic, inverse-trigonometric, algebraic, trigonometric, exponential. The function earlier in the list is often a good choice for u .

Worked example 5: integration by parts

Evaluate

$$\int_0^1 x e^x dx.$$

Choose $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Thus

$$\int_0^1 x e^x dx = [x e^x]_0^1 - \int_0^1 e^x dx = e - (e - 1) = 1.$$

Checkpoint: Checkpoint

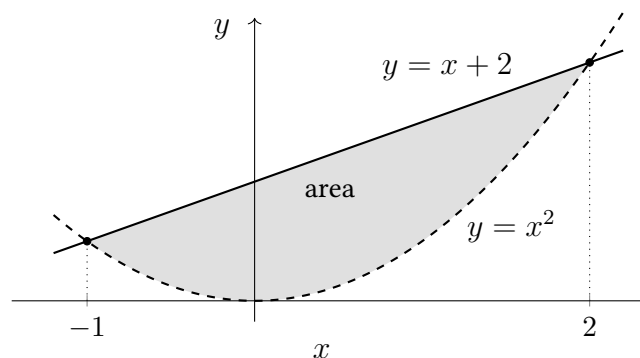
In substitution, the main question is “what inner function has its derivative nearby?” In integration by parts, the main question is “which part becomes simpler after differentiating?”

5 Modeling applications of definite integrals

Area between two curvesIf $f(x) \geq g(x)$ on $[a, b]$, the area between the curves is

$$A = \int_a^b [f(x) - g(x)] dx.$$

The order matters: top curve minus bottom curve.



$$\text{Area} = \int_{-1}^2 [(x + 2) - x^2] dx$$

Figure 4: For area between curves, integrate upper minus lower. The shaded region is bounded by $y = x + 2$ and $y = x^2$ from $x = -1$ to $x = 2$.

Worked example 6: area between curvesFind the area enclosed by $y = x^2$ and $y = x + 2$.Intersections satisfy $x^2 = x + 2$, so $x = -1$ or $x = 2$. On this interval, $x + 2 \geq x^2$. Hence

$$A = \int_{-1}^2 [(x + 2) - x^2] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}.$$

Average value and accumulated change

For a continuous function f on $[a, b]$, its **average value** is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

If $Q'(t) = r(t)$, then

$$Q(b) - Q(a) = \int_a^b r(t) dt.$$

This is the form used in flow, pollution, population, inventory, and energy models.

Worked example 7: average inventory

In the EOQ model, suppose inventory decreases linearly from Q to 0 over a cycle of length T :

$$I(t) = Q \left(1 - \frac{t}{T} \right), \quad 0 \leq t \leq T.$$

The average inventory is

$$\bar{I} = \frac{1}{T} \int_0^T Q \left(1 - \frac{t}{T} \right) dt = \frac{Q}{T} \left[t - \frac{t^2}{2T} \right]_0^T = \frac{Q}{2}.$$

This familiar result is an integral statement, not only a geometric triangle argument.

Model criticism: Model criticism: units and sign

Before reporting an integral, check its units. If $r(t)$ is litres/hour and t is measured in hours, then $\int r(t) dt$ is litres. Also check whether signed area makes sense: negative values may represent out-flow, loss, or correction rather than literal negative area.

6 Improper integrals and convergence

Some modeling integrals extend over an infinite interval or include an unbounded integrand. These are **improper integrals**. They must be defined using limits.

Two common improper integrals

For an infinite interval,

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

For a vertical blow-up at a ,

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

If the limit exists, the integral **converges**. Otherwise it **diverges**.

Worked example 8: convergence versus divergence

Compare

$$\int_1^{\infty} \frac{1}{x^2} dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x} dx.$$

For the first,

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1,$$

so it converges. For the second,

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln b = \infty,$$

so it diverges. Infinite interval does not automatically mean infinite area.

Checkpoint: Checkpoint

Why is the computation $\int_{-1}^1 1/x^2 dx = [-1/x]_{-1}^1$ invalid? Identify the hidden discontinuity.

7 Numerical integration

Many integrals in modeling do not have elementary antiderivatives. For example, $\int e^{-x^2} dx$ appears in statistics, but cannot be expressed using elementary functions. We then approximate.

Trapezoidal and Simpson rules

For a regular partition of $[a, b]$ with n subintervals and $h = (b - a)/n$:

$$T_n = \frac{h}{2} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

If n is even, Simpson's rule is

$$S_n = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

Trapezoidal uses straight-line panels; Simpson uses quadratic panels.

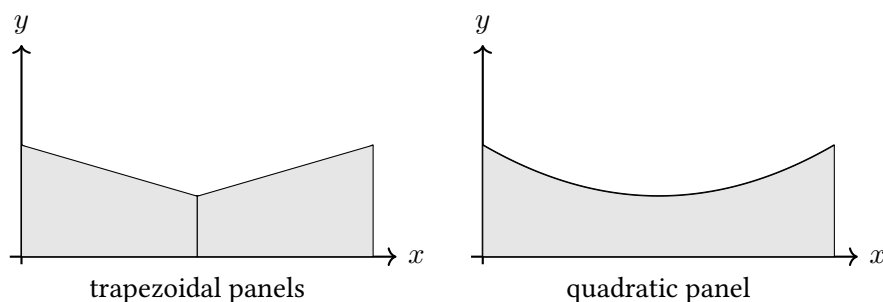


Figure 5: Trapezoidal panels approximate the curve by straight segments. Simpson's rule approximates over pairs of subintervals by a quadratic curve.

Worked example 9: comparing numerical rules

Estimate

$$\int_0^1 e^{-x^2} dx$$

using $n = 4$ subintervals. With $h = 0.25$, the function values are approximately

x_k	0	0.25	0.50	0.75	1.00
$e^{-x_k^2}$	1.00000	0.93941	0.77880	0.56978	0.36788

Thus

$$T_4 = 0.74298, \quad S_4 = 0.74685.$$

The true value is approximately 0.746824, so Simpson's rule is much more accurate here.

Model criticism: Numerical integration in a modeling report

A good report should not only state “we used Simpson's rule.” It should specify the interval, step size, stopping criterion, and a validation check. For example:

We estimated the integral using Simpson's rule with step size $h = 0.01$. Doubling the number of subintervals changed the value by less than 10^{-4} , so numerical error is small relative to data uncertainty.

8 Summary and practice

Key ideas

- Riemann sums approximate integrals by finite pieces.
- The definite integral is an accumulation operator: area, total change, average value, probability, or mass.
- FTC connects accumulated change and instantaneous rate.
- Substitution is reverse chain rule; integration by parts is reverse product rule.
- Improper integrals require limits and may converge or diverge.
- Numerical integration is a modeling choice; accuracy and cost should be justified.

Core practice

1. Compute left, right, and midpoint sums for $\int_0^2 x^2 dx$ with $n = 4$. Compare with the exact value.
2. Evaluate $\int_1^4 (3x^2 - 2x + 5) dx$ using the FTC.
3. Evaluate $\int_0^1 2x(1 + x^2)^5 dx$ using substitution.
4. Evaluate $\int_1^e \ln x dx$ using integration by parts.
5. Determine whether $\int_1^\infty x^{-3/2} dx$ converges.
6. Find the average value of $f(x) = x^2$ on $[0, 3]$.

Practice check and hints

1. The left, right, and midpoint sums are $7/4$, $15/4$, and $21/8$; the exact value is $8/3$.
2. The FTC gives 63.
3. With $u = 1 + x^2$, the integral is $21/2$.
4. Integration by parts gives 1.
5. The improper integral converges to 2.
6. The average value is 3.

Modeling challenge: cafeteria inflow

During the first 30 minutes of lunch, the arrival rate of students is modeled by

$$r(t) = 18 + 0.9t - 0.03t^2, \quad 0 \leq t \leq 30,$$

where t is measured in minutes and $r(t)$ in students per minute.

1. Compute the total number of students arriving in the first 30 minutes.
2. Find the average arrival rate.
3. Explain one assumption behind using a smooth rate function for student arrivals.
4. Suggest one way to validate the model using observed queue data.

Modeling checks

The model predicts 675 arrivals and an average rate of 22.5 students per minute. A complete response also distinguishes a smooth expected rate from discrete arrivals, names a data-collection interval and measurement rule, and proposes an out-of-sample comparison between integrated predictions and observed counts.

Extension practice

1. Show that if f is odd and integrable on $[-a, a]$, then $\int_{-a}^a f(x) dx = 0$.
2. Use Simpson's rule with $n = 4$ to approximate $\int_0^1 \frac{1}{1+x^2} dx$ and compare with $\pi/4$.
3. Verify that $\int_0^\infty \lambda e^{-\lambda x} dx = 1$ for $\lambda > 0$, and explain why this matters for probability density models.

Extension hints

For Question 1, substitute $u = -x$ on one half of the interval. For Question 2, Simpson's rule gives $8011/10200 \approx 0.785392$, close to $\pi/4 \approx 0.785398$. For Question 3, evaluate the truncated integral first and then let its upper limit tend to infinity; the value is 1, as required for a probability density.

Model audit card

Audit before trusting or communicating a model			
Purpose	What question or decision does the model support?	Data	Which inputs are observed, derived, or simulated?
Assumptions	Which simplifications may change the answer?	Units	Are dimensions and scales consistent?
Verification	Were equations, code, constraints, and limiting cases checked?	Validation	Is performance tested against independent evidence?
Uncertainty	How do measurement, parameter, solver, and structural errors affect conclusions?	Ethics	Who benefits, who bears risk, and what fairness criterion matters?
Reproducibility	Are data provenance, versions, seeds, and transformations recorded?	Communication	Are claims conditional, traceable, and decision-relevant?

Further reading

- Calculus Volume 1 (OpenStax)
- Calculus Volume 2 (OpenStax)

Sources, provenance, and licence

These notes follow the iterative modeling and assessment perspective in the SIAM/COMAP GAIMME report. Competition-specific remarks are a snapshot checked on 20 August 2026 against the official COMAP HiMCM instructions; readers should verify current rules before submission. Unless a source is explicitly identified, classroom datasets and numerical scenarios are synthetic or simulated teaching examples and are not empirical claims about a real institution. Third-party references retain their own terms. Original teaching material by Kenneth Sok Kin Cheng is released under CC BY-NC-SA 4.0.

Student response guide and checking criteria

For closed calculations, show the definition, substitution, units, and at least one independent check; the worked examples in this unit are method hints, but you must complete the arithmetic yourself. Open modeling responses are checked on five dimensions: a clear purpose and output; stated data and provenance; defensible assumptions and scope; traceable computational verification and model validation; and alignment among uncertainty, limitations, and the recommendation. Support each dimension with concrete evidence rather than writing only “completed.”