

Lectures in Mathematical Modeling

Supplementary 5: Derivatives, Rates, and Local Linear Models

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Learning goals

By the end of this supplementary lesson, readers should be able to:

1. interpret the **derivative** as a local slope, instantaneous rate, and sensitivity coefficient;
2. compute simple derivatives from the **difference quotient**;
3. use the power, product, quotient, trigonometric, exponential, logarithmic, and **chain rule** correctly;
4. identify common reasons why a function is not differentiable;
5. use implicit, parametric, and related-rate differentiation in geometric or physical models;
6. use the **Mean Value Theorem** to connect average and instantaneous rates;
7. explain derivative calculations in a modeling report instead of presenting them as isolated algebra.

1 Derivative as slope, rate, and sensitivity

In modeling, a derivative is not just a calculus operation. It answers a practical question: *how fast does the output change when the input changes slightly?* If $y = f(x)$, then $f'(a)$ measures the local change in y per unit change in x near $x = a$.

Launch: average speed versus speed at one instant

[12 min]

A robot travels along a corridor. Its distance from the start after t seconds is modeled by

$$s(t) = 0.4t^2 + 1.2t \quad (0 \leq t \leq 10),$$

where s is in metres.

1. Compute the average speed from $t = 3$ to $t = 5$.
2. Compute the average speed from $t = 4$ to $t = 4.1$.
3. What value should the speedometer display at exactly $t = 4$?

Discussion point: smaller time windows give more local information. The derivative is the limiting value of these average rates.

Core definition

For a function f , the **derivative at a** is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. The fraction is the **difference quotient**. It is the slope of the secant line through $(a, f(a))$ and $(a + h, f(a + h))$.

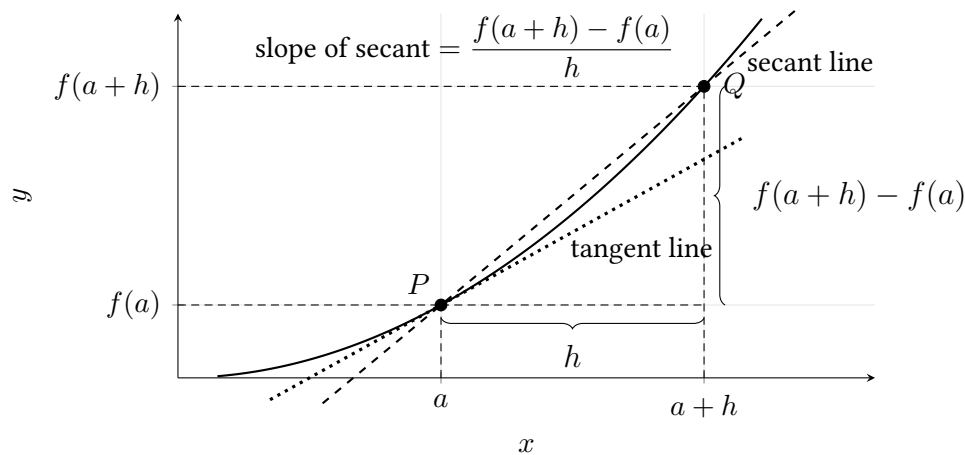


Figure 1: The derivative at a is the limit of the slopes of secant lines: $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$. As h becomes smaller, the secant line through P and Q approaches the tangent line at P .

Worked example 1: speed from a position model

For $s(t) = 0.4t^2 + 1.2t$, the instantaneous speed at $t = 4$ is

$$s'(4) = \lim_{h \rightarrow 0} \frac{s(4+h) - s(4)}{h}.$$

Compute

$$\begin{aligned} s(4+h) &= 0.4(4+h)^2 + 1.2(4+h) \\ &= 0.4(16 + 8h + h^2) + 4.8 + 1.2h \\ &= 11.2 + 4.4h + 0.4h^2, \end{aligned}$$

while $s(4) = 11.2$. Hence

$$\frac{s(4+h) - s(4)}{h} = 4.4 + 0.4h \rightarrow 4.4.$$

So the robot's speed at $t = 4$ is 4.4 m/s.

Checkpoint: Checkpoint 1

Explain the units of $s'(4)$. Why is it not measured in metres?

2 First-principles derivatives

Before using rules, students should experience why the rules are true. This prevents derivatives from becoming a memorised table detached from modeling.

Worked example 2: derivative of x^2 from the definition

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

At $x = 3$, the local slope is 6. This means that near $x = 3$, increasing x by 0.01 increases x^2 by about 0.06.

Worked example 3: derivative of $\sqrt{x}$

For $x > 0$,

$$\begin{aligned}\frac{d}{dx}\sqrt{x} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.\end{aligned}$$

The derivative is larger near $x = 0$, so the square-root graph is steep near the origin and gradually flattens.

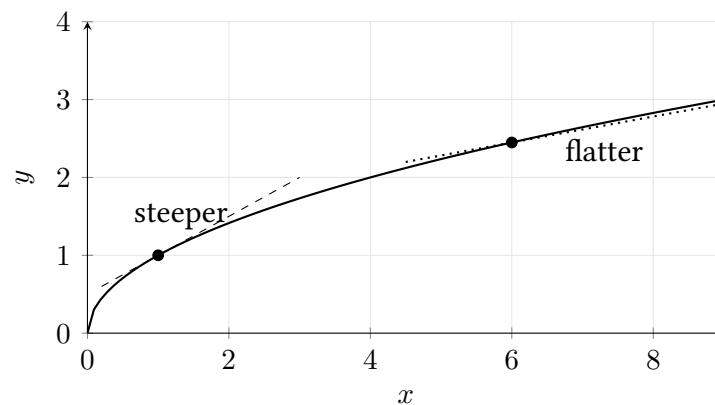


Figure 2: The derivative of $\sqrt{x}$ decreases as x increases. This is a useful visual interpretation of $1/(2\sqrt{x})$.

Model criticism: Model critique: derivative as local linear model

A derivative gives a local approximation:

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x.$$

This is powerful, but it is local. It may be poor when Δx is large or when the graph bends sharply. In a modeling report, always state the interval over which the local approximation is being used.

3 Differentiation rules for efficient calculation

Core derivative rules

If f and g are differentiable, then

$$\begin{aligned}\frac{d}{dx}(c) &= 0, & \frac{d}{dx}(x^n) &= nx^{n-1}, & \frac{d}{dx}[cf] &= cf', \\ \frac{d}{dx}(f+g) &= f' + g', & \frac{d}{dx}(fg) &= f'g + fg', & \frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{f'g - fg'}{g^2}.\end{aligned}$$

The product rule is not $(fg)' = f'g'$. The quotient rule is not the derivative of numerator divided by derivative of denominator.

Worked example 4: product and quotient rules

Differentiate

$$F(x) = x^2 \sin x + \frac{x^2 + 1}{x - 1}.$$

For the product term,

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x.$$

For the quotient term,

$$\frac{d}{dx} \left(\frac{x^2 + 1}{x - 1} \right) = \frac{2x(x - 1) - (x^2 + 1)}{(x - 1)^2} = \frac{x^2 - 2x - 1}{(x - 1)^2}.$$

Therefore

$$F'(x) = 2x \sin x + x^2 \cos x + \frac{x^2 - 2x - 1}{(x - 1)^2}.$$

Worked example 5: trigonometric and exponential derivatives

For a seasonal demand model

$$D(t) = 120 + 30 \sin \left(\frac{2\pi t}{12} \right) + 5e^{0.03t},$$

where t is measured in months, compute the instantaneous rate of change:

$$D'(t) = 30 \cos \left(\frac{2\pi t}{12} \right) \cdot \frac{2\pi}{12} + 5e^{0.03t} \cdot 0.03.$$

Thus

$$D'(t) = 5\pi \cos \left(\frac{\pi t}{6} \right) + 0.15e^{0.03t}.$$

The first term describes seasonal increase/decrease; the second term describes long-term growth.

Checkpoint: Checkpoint 2

Why is the derivative of $\sin(\pi t/6)$ not simply $\cos(\pi t/6)$? Name the rule being used.

4 The chain rule as a modeling pipeline

The **chain rule** is essential for modeling because many models are pipelines: an input affects an intermediate quantity, which affects the output.

Chain rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In function notation,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

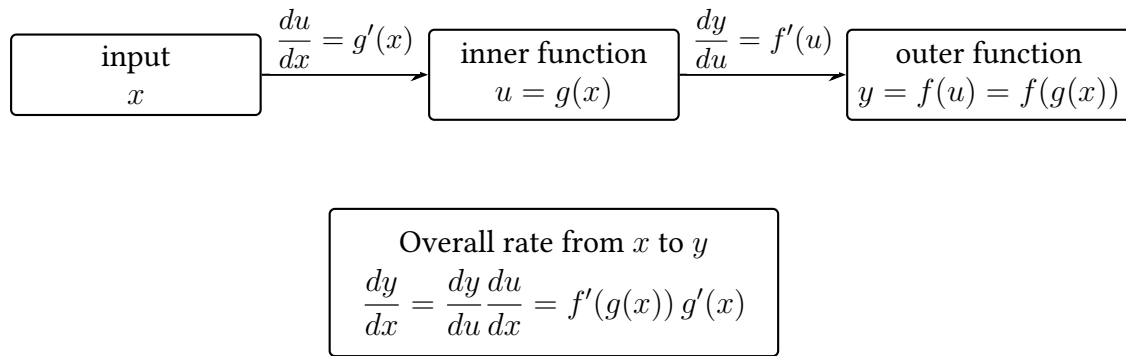


Figure 3: The chain rule for a composite function $y = f(g(x))$. The input x first changes the intermediate variable $u = g(x)$, and then u changes the final output $y = f(u)$. Thus the overall rate is the product of the two rates.

Worked example 6: pollution concentration through a transformation

A concentration index is modeled by

$$I(t) = \sqrt{1 + 0.4C(t)}, \quad C(t) = 20e^{-0.08t}.$$

Find $I'(t)$ and interpret the sign.

$$I'(t) = \frac{1}{2\sqrt{1 + 0.4C(t)}} \cdot 0.4C'(t).$$

Since $C'(t) = -1.6e^{-0.08t}$,

$$I'(t) = \frac{0.2(-1.6e^{-0.08t})}{\sqrt{1 + 8e^{-0.08t}}} < 0.$$

The index is decreasing. The derivative becomes closer to zero over time because the exponential decay slows in absolute size.

Worked example 7: logarithmic differentiation

Differentiate $y = x^x$ for $x > 0$.

$$\ln y = x \ln x.$$

Differentiate both sides:

$$\frac{y'}{y} = \ln x + 1.$$

Therefore

$$y' = x^x (\ln x + 1).$$

This method is useful when both the base and exponent vary.

5 When derivatives do not exist

A derivative may fail to exist even for a simple-looking graph. In a model, that failure may mark a switch, threshold, collision, capacity boundary, or policy change.

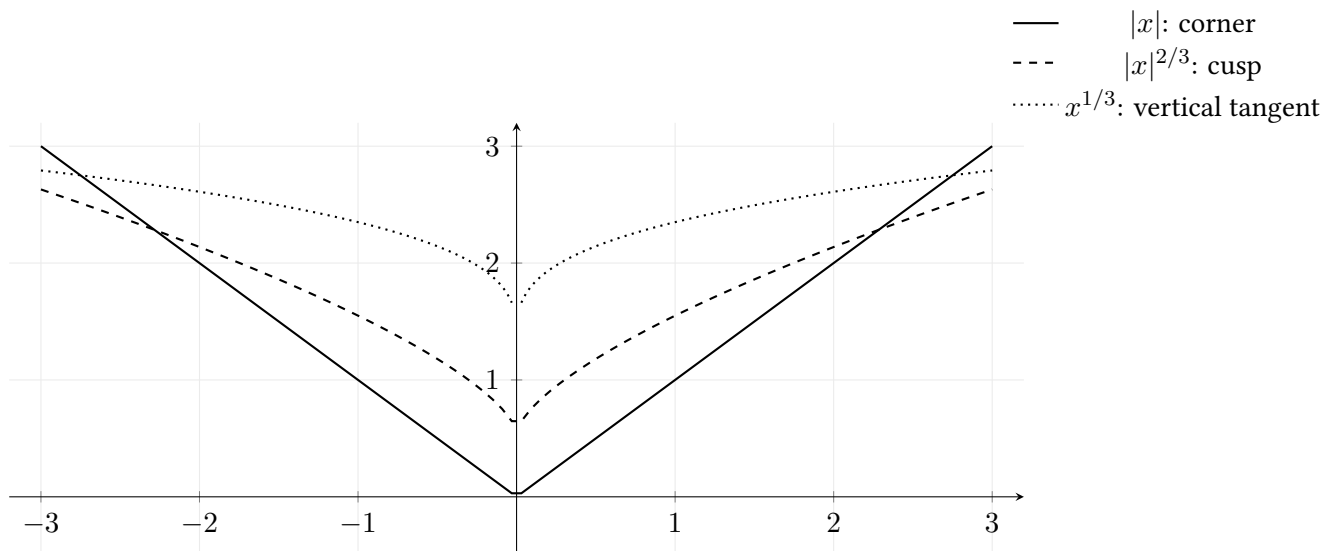


Figure 4: Three common non-differentiable behaviours. Each can have a modeling interpretation.

Checkpoint: Checkpoint 3

For $f(x) = |x|$, compute the right-hand derivative and left-hand derivative at 0. Why does the derivative not exist there?

Model criticism: Model critique: smoothing versus preserving a threshold

In applications, replacing a kink by a smooth curve may make calculus easier, but it may erase a real threshold. For example, a penalty fee may start exactly when demand exceeds capacity. A good modeler should ask: is the corner a mathematical inconvenience, or is it the phenomenon we need to preserve?

6 Implicit, parametric, and related-rate differentiation

Not all models are naturally written as $y = f(x)$. Circles, ellipses, supply-demand equilibria, and geometric constraints often appear as equations involving several variables.

Worked example 8: implicit differentiation on a circle

For a circle

$$x^2 + y^2 = 25,$$

we differentiate both sides with respect to x :

$$2x + 2y \frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

At $(3, 4)$, the tangent slope is $-3/4$, so the tangent line is

$$y - 4 = -\frac{3}{4}(x - 3).$$

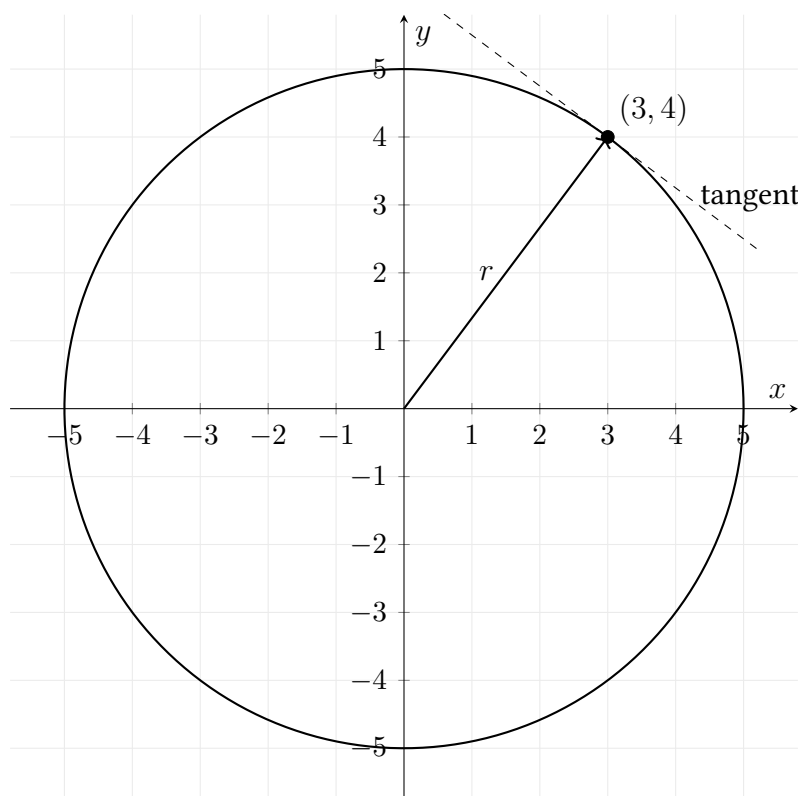


Figure 5: Implicit differentiation finds tangent slopes even when the curve is not a function of x globally.

Parametric derivative

If a curve is given by $x = g(t)$ and $y = h(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad dx/dt \neq 0.$$

This is useful for motion models because t is the natural independent variable.

Worked example 9: related rates for a sliding ladder

A 10 m ladder leans against a wall. Its base moves away at 0.5 m/s. How fast is the top moving down when the base is 6 m from the wall?

Let x be the distance of the base from the wall and y the height of the top. Then

$$x^2 + y^2 = 100.$$

Differentiate with respect to time:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

When $x = 6$, $y = 8$. Hence

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{6}{8}(0.5) = -0.375 \text{ m/s}.$$

The negative sign means the height is decreasing.

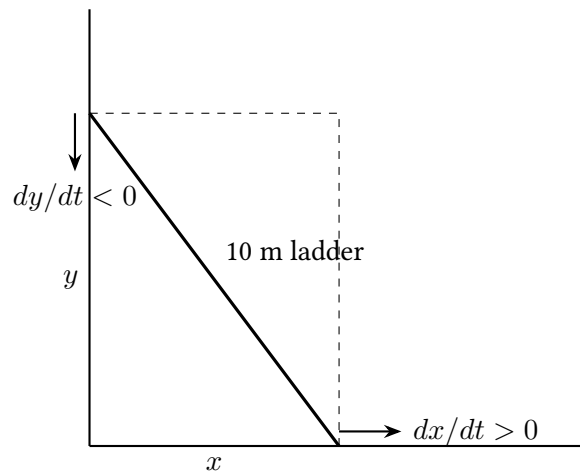


Figure 6: Related rates translate a geometric constraint into a relationship between rates.

7 Mean Value Theorem and local linear thinking

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The instantaneous rate somewhere equals the average rate over the interval.

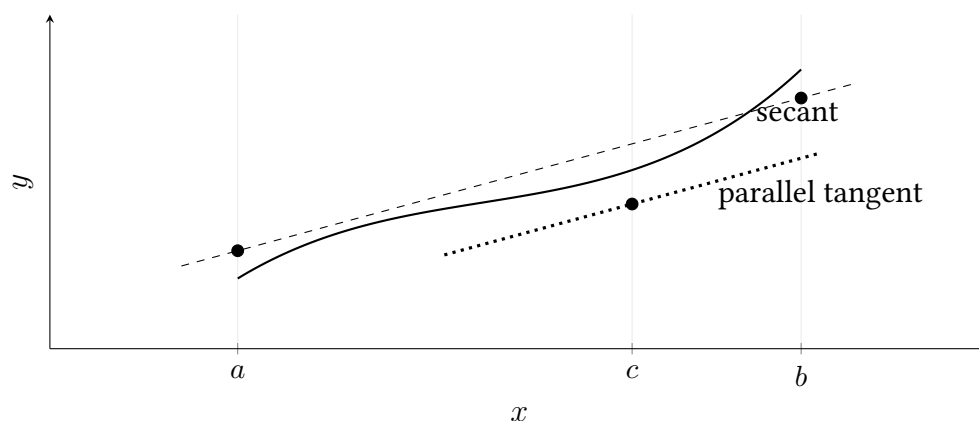


Figure 7: The Mean Value Theorem links an average rate to at least one instantaneous rate.

Worked example 10: interpreting an average-velocity claim

A car's position along a straight route increases by 120 km over 1.5 hours. If its position function is continuous on the time interval and differentiable in the interior, the Mean Value Theorem says that at some instant its velocity was exactly

$$\frac{120}{1.5} = 80 \text{ km/h.}$$

This does not say the velocity was always 80 km/h. It says that an instantaneous velocity equal to the average velocity occurred at least once. If the car moved only forwards, its speed at that instant was also 80 km/h.

What students should take away

- A derivative is a local rate, slope, and sensitivity.
- The difference quotient is the bridge from average rate to instantaneous rate.
- Rules speed up calculation, but the chain rule is the main rule for composite models.
- Non-differentiability is often meaningful, not merely a technical failure.
- Implicit and related-rate methods are essential when the model is constrained geometrically.
- The MVT connects local and average behaviour, which is why derivatives support real-world interpretation.

8 Optional extension: L'Hôpital's Rule

Reserve material

L'Hôpital's Rule belongs after students understand derivatives. If a limit has indeterminate form $0/0$ or ∞/∞ , and the relevant derivative quotient limit exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

It should not be used before checking the indeterminate form.

Optional example: exponential dominates polynomial

Evaluate

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x}.$$

The form is ∞/∞ . Applying L'Hôpital twice gives

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} \rightarrow \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0.$$

Thus exponential growth eventually dominates quadratic growth.

9 Exercises

Core practice

1. Use the definition of derivative to find the derivative of $f(x) = 3x^2 - 2x$.
2. Differentiate:

$$f(x) = 5x^4 - 3x^{-2} + 7\sqrt{x} - 9.$$
3. Differentiate $g(x) = (x^2 + 1)(\sin x + x)$.
4. Differentiate $h(x) = \frac{x^2 + 3x}{x - 2}$.
5. Differentiate $p(x) = \sin(3x^2 + 1)$ and explain which part is the inner function.
6. Find the tangent line to $x^2 + y^2 = 13$ at $(2, 3)$.

Practice check and hints

1. The derivative from the definition is $6x - 2$.
2. The result is $20x^3 + 6x^{-3} + 7/(2\sqrt{x})$ on the original real domain.
3. Use the product rule to obtain $2x(\sin x + x) + (x^2 + 1)(\cos x + 1)$.
4. The quotient-rule result simplifies to $(x^2 - 4x - 6)/(x - 2)^2$.
5. The inner function is $3x^2 + 1$, and $p'(x) = 6x \cos(3x^2 + 1)$.
6. The tangent has slope $-2/3$: $y - 3 = -\frac{2}{3}(x - 2)$.

Modeling practice

1. A queue waiting-time model is $W(\rho) = \frac{4\rho}{1 - \rho}$ for $0 < \rho < 1$. Compute $W'(\rho)$ and explain what happens as $\rho \rightarrow 1^-$. Why is this important for capacity planning?
2. A pollutant concentration follows $C(t) = 60e^{-0.12t} + 8$. Compute $C'(t)$ and determine when the magnitude of the rate first falls below 1 unit/day.
3. A circular infection front has radius $r(t)$ and area $A(t) = \pi r(t)^2$. If $dr/dt = 0.3$ km/day, find dA/dt when $r = 5$ km.
4. A seasonal sales model is $S(t) = 200 + 40 \cos(\pi t/6)$. Find when sales are increasing fastest and interpret your answer.

Modeling checks

For Question 1, $W'(\rho) = 4/(1 - \rho)^2$, which diverges as $\rho \rightarrow 1^-$. For Question 2, $|C'(t)|$ crosses 1 at $t = \ln(7.2)/0.12 \approx 16.45$ days and is below 1 thereafter. Question 3 gives $dA/dt = 3\pi$ km²/day. In Question 4, the fastest increase occurs at $t = 9 + 12k$; within one 12-month cycle, use $t = 9$. Include units, the stated domain, and a decision-relevant interpretation.

Challenge and extension

1. Show that if $f'(x) = 0$ for all x in an interval, then f is constant on that interval. State clearly where the Mean Value Theorem is used.
2. Differentiate $y = x^x \sin x$ for $x > 0$.
3. For the parametric curve $x = t^2 + 1$, $y = t^3 - t$, find dy/dx and the slope at $t = 2$.
4. Use L'Hôpital's Rule to evaluate

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$$

5. Write a short modeling paragraph explaining why a derivative-based approximation may fail near a kink or threshold.

Challenge checks

For Question 1, apply the Mean Value Theorem on any closed subinterval of the given interval. For Question 2, $y' = x^x[(\ln x + 1) \sin x + \cos x]$ for $x > 0$. For Question 3, $dy/dx = (3t^2 - 1)/(2t)$ for $t \neq 0$, and the slope at $t = 2$ is $11/4$. Question 4 has limit $1/2$. Question 5 should identify the kink or threshold, explain why one local slope cannot cross it, and suggest a piecewise or one-sided model.

Model audit card

Audit before trusting or communicating a model			
Purpose	What question or decision does the model support?	Data	Which inputs are observed, derived, or simulated?
Assumptions	Which simplifications may change the answer?	Units	Are dimensions and scales consistent?
Verification	Were equations, code, constraints, and limiting cases checked?	Validation	Is performance tested against independent evidence?
Uncertainty	How do measurement, parameter, solver, and structural errors affect conclusions?	Ethics	Who benefits, who bears risk, and what fairness criterion matters?
Reproducibility	Are data provenance, versions, seeds, and transformations recorded?	Communication	Are claims conditional, traceable, and decision-relevant?

Further reading

- Calculus Volume 1 (OpenStax)

Sources, provenance, and licence

These notes follow the iterative modeling and assessment perspective in the SIAM/COMAP GAIMME report. Competition-specific remarks are a snapshot checked on 20 August 2026 against the official COMAP HiMCM instructions; readers should verify current rules before submission. Unless a source is explicitly identified, classroom datasets and numerical scenarios are synthetic or simulated teaching examples and are not empirical claims about a real institution. Third-party references retain their own terms. Original teaching material by Kenneth Sok Kin Cheng is released under CC BY-NC-SA 4.0.

Student response guide and checking criteria
For closed calculations, show the definition, substitution, units, and at least one independent check; the worked examples in this unit are method hints, but you must complete the arithmetic yourself. Open modeling responses are checked on five dimensions: a clear purpose and output; stated data and provenance; defensible assumptions and scope; traceable computational verification and model validation; and alignment among uncertainty, limitations, and the recommendation. Support each dimension with concrete evidence rather than writing only “completed.”