Lectures in Mathematical Modeling

Supplementary 3: Graphs, Transformations, and Periodic Models

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Learning goals

By the end of this supplementary lesson, readers should be able to:

1. read a **graph** as a model of a relationship between two quantities;
2. distinguish a **relation** from a **function** using the vertical-line test;
3. determine **domain**, **range**, intercepts, symmetry, and key features from formulas and graphs;
4. use graph transformations to sketch related functions without re-plotting from scratch;
5. interpret elementary graph families as modeling shapes: linear, quadratic, reciprocal, absolute value, square-root, and sinusoidal;
6. build a simple **periodic model** using amplitude, period, phase shift, and vertical shift;
7. explain what a graph reveals, what it hides, and what assumptions are needed before using it for prediction.

1 Graphs as modeling language

A graph is not merely a picture of a formula. In mathematical modeling, a graph is a compact way to communicate a relationship: increase, decrease, saturation, threshold, periodicity, outliers, and breakdown of assumptions can often be seen before they are calculated.

Launch: what can a graph say? **[12 min]**

A school records the waiting time at a canteen from 12:00 to 12:40. The curve rises quickly, reaches a peak, then slowly returns to nearly zero.

1. What does the peak mean in context?
2. What would a flatter curve mean?
3. What information is missing if we only know the curve shape but not the units?
4. What intervention could move the peak downward or earlier?

Modeling lesson: a graph should be read as a statement about a system, not only as a drawing.

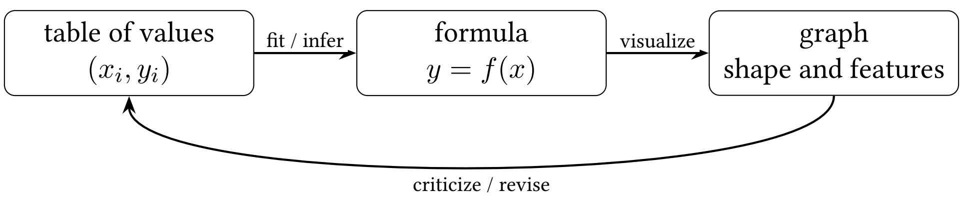

Figure 1: In modeling, data, formula, and graph should support one another.

Core vocabulary

A **relation** is a set of ordered pairs (x, y). A **function** is a special relation in which each input x corresponds to exactly one output y. The **domain** is the set of allowed inputs; the **range** is the set of outputs produced. The graph of $y = f(x)$ is the set of all points $(x, f(x))$.

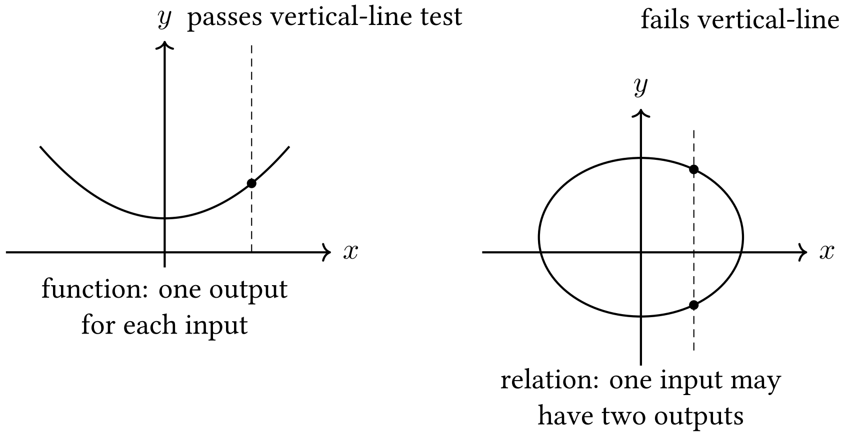

Figure 2: The vertical-line test checks whether a relation can be written as $y = f(x)$.

2 Domain, range, intercepts, and symmetry

Before fitting or interpreting a graph, a modeler must decide which values are meaningful. In school mathematics, a formula may have a natural algebraic domain. In modeling, the realistic domain may be smaller.

Worked Example 1: Natural domain versus modeling domain

A simplified stopping-distance model is

$$D(v) = 0.75v + 0.006v^2,$$

where v is speed in km/h and D is stopping distance in metres.
Algebraic domain: the formula accepts all real v. **Modeling domain**: speed cannot be negative, and the model was calibrated only for ordinary road speeds, say $0 \le v \le 120$.
The y-intercept is $D(0) = 0$, meaning no stopping distance when speed is zero. There is no meaningful negative-speed branch, even though the algebraic formula could be evaluated there. The graph should therefore be drawn only on the modeling domain.

Worked Example 2: Domain and range from a formula

Find the natural domain and range of

$$f(x) = \sqrt{4 - x^2}.$$

The square root requires $4 - x^2 \ge 0$, so $-2 \le x \le 2$. Hence

$$\mathrm{Dom}(f) = [-2, 2].$$

Since $f(x)$ is the upper half of the circle $x^2 + y^2 = 4$, the output satisfies $0 \le y \le 2$:

$$\mathrm{Range}(f) = [0, 2].$$

This is a function because each x in $[-2, 2]$ gives exactly one nonnegative y.

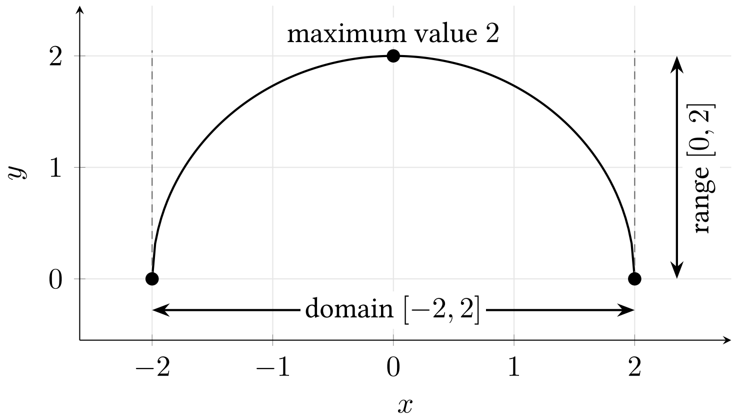

Figure 3: The graph of $y = \sqrt{4 - x^2}$ is the upper semicircle of radius 2, so its domain and range are visible.

Checkpoint: Feature check

For each function, state the natural domain, one intercept, and whether the graph is even, odd, or neither.

1. $f(x) = x^4 - 3x^2$
2. $g(x) = \dfrac{1}{x-2}$
3. $h(x) = \sqrt{x+1}$

3 Graph transformations and standard shapes

Graph transformations allow us to sketch related functions quickly. This is important in modeling because fitted or theoretical curves often differ from a standard graph only by scaling or shifting.

Transformations of $y = f(x)$

New function	Effect on the graph
$y = f(x) + c$	shift upward by c
$y = f(x - c)$	shift right by c
$y = af(x)$	vertical stretch by factor $\lvert a \rvert$; reflect in x-axis if $a < 0$
$y = f(bx)$	horizontal compression by factor b if $b > 1$
$y = -f(x)$	reflect in the x-axis
$y = f(-x)$	reflect in the y-axis

Worked Example 3: Transforming a graph

Describe the graph of

$$y = -2\sqrt{x+3} + 4.$$

Start with $y = \sqrt{x}$.

1. $\sqrt{x+3}$ shifts the graph left by 3.
2. $-2\sqrt{x+3}$ reflects it in the x-axis and stretches vertically by factor 2.

3. $-2\sqrt{x+3}+4$ shifts the graph upward by 4.
The starting point moves from $(0,0)$ to $(-3,4)$. The domain is $[-3,\infty)$ and the range is $(-\infty,4]$.

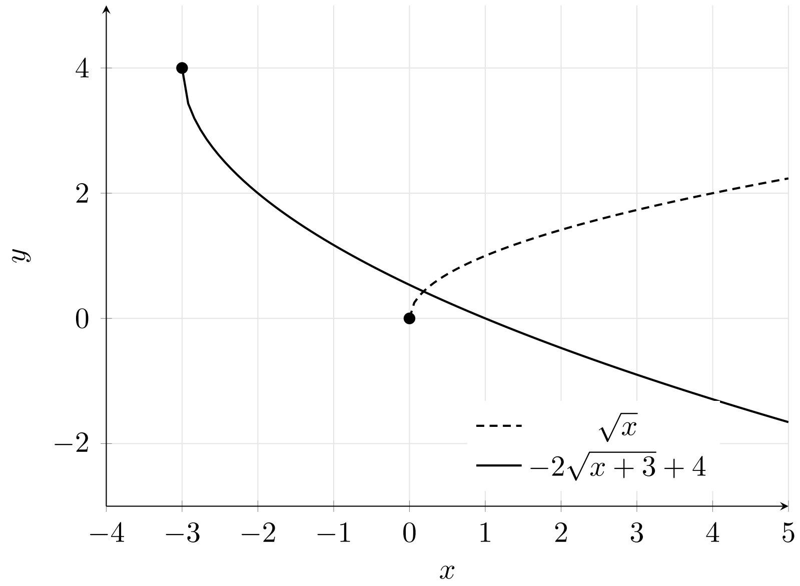

Figure 4: Transformations let us sketch a new model from a familiar parent graph.

Standard graph families for modeling

Family	Typical formula	Modeling meaning
Linear	$mx+b$	constant rate of change; calibration line
Quadratic	ax^2+bx+c	accelerating change, projectile-like shapes, cost curves
Absolute value	$a\lvert x-h\rvert+k$	distance from a target, penalty around a desired value
Square root	$a\sqrt{x-h}+k$	fast initial growth then slowing; diminishing gains
Reciprocal	$a/(x-h)+k$	inverse relationship, asymptote, capacity or congestion effects
Sinusoidal	$A\sin(B(t-C))+D$	periodic or seasonal behaviour

Model criticism: Choosing a graph family is an assumption

If a student chooses a quadratic graph only because it fits six points well, the model may extrapolate badly. A graph family should be justified by a mechanism: constant rate, saturation, inverse dependence, periodic forcing, or geometric constraint. A competition report should state why the chosen shape is meaningful, not only that it fits the data.

4 Trigonometric graphs as periodic models

The most useful part of trigonometry for modeling is not memorizing identities; it is recognizing repeated behaviour. Many systems have a repeating pattern: daylight, tides, traffic cycles, temperature, electricity demand, and school attendance.

Sinusoidal model

A basic periodic model has the form

$$y = A\sin(B(t - C)) + D.$$

Here $|A|$ is the **amplitude**, $2\pi/|B|$ is the **period**, C is the **phase shift**, and D is the **midline**. The same interpretation applies to cosine models.

Worked Example 4: A seasonal attendance model

Suppose the average number of students absent from a school follows a yearly cycle. The minimum is about 8 students in September, the maximum is about 28 students in March, and the period is one year. Let t be months after September.
The midline is

$$D = \frac{28+8}{2} = 18,$$

and the amplitude is

$$A = \frac{28-8}{2} = 10.$$

Because the minimum occurs at $t = 0$, a convenient cosine model is

$$\boxed{A(t) = 18 - 10\cos\left(\frac{2\pi}{12}t\right).}$$

At $t = 6$, the model gives $A(6) = 28$, representing the early-spring peak in March. This model should undergo **validation** against several years of records before being used for staffing decisions.

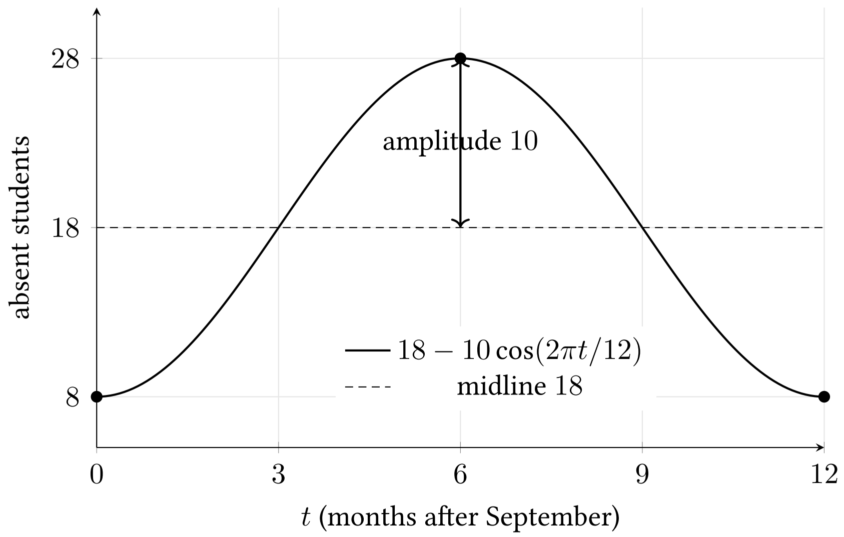

Figure 5: A sinusoidal graph turns a repeating pattern into interpretable parameters.

Checkpoint: Periodic model check

A tide height ranges from 1.2 m to 3.8 m and repeats every 12 hours. At $t = 0$, the tide is at its maximum.

1. Find the amplitude and midline.

2. Write a cosine model.

3. Explain one limitation of this model.

5 Asymptotes and conics as reserve modeling shapes

Some graphs are useful because they show boundaries rather than ordinary turning points. An **asymptote** describes a value the curve approaches but does not cross in a simple way. A **conic section** describes geometric constraints such as fixed distance, fixed sum of distances, or fixed difference of distances.

Worked Example 5: Reading asymptotes in a congestion model

Suppose a simple travel-time model is

$$T(q) = 12 + \frac{40}{100 - q}, \qquad 0 \leq q < 100,$$

where q is the number of vehicles entering a short road segment per minute. The graph has a vertical asymptote at $q = 100$. As q approaches capacity, the predicted travel time increases rapidly. The asymptote is a warning: operating close to capacity is unstable and small demand changes can cause large delays.

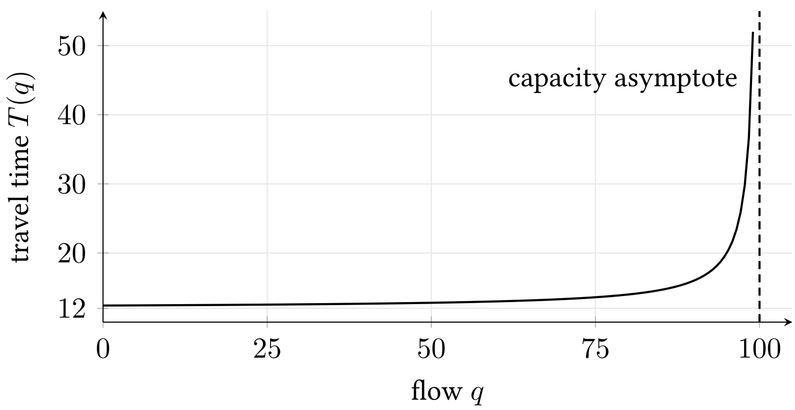

Figure 6: An asymptote can represent a practical capacity boundary.

Conics in one paragraph

Conic	Defining idea	Modeling use
Circle	fixed distance from a centre	coverage, safety radius, sensor range
Parabola	equal distance from focus and directrix	projectile shape, reflector geometry
Ellipse	fixed sum of distances to two foci	orbit-like shapes, service regions
Hyperbola	fixed difference of distances to two foci	localization from time-difference signals

6 Writing graph-based modeling conclusions

Model criticism: Competition-style communication

A graph-based conclusion should include four parts:

1. **Feature:** What graph property matters? For example: peak, intercept, slope, asymptote, period.
2. **Meaning:** What does that feature represent in the real system?
3. **Decision:** What action follows from this interpretation?
4. **Limitation:** What would make the graph unreliable?

Weak conclusion: "The graph goes up and then down."
Stronger conclusion: "The waiting-time curve peaks at about 12:18, so the canteen bottleneck is concentrated in the first half of lunch. A second service counter should be opened before the peak rather than after it. This conclusion assumes that the recorded day is typical and that waiting time was measured consistently."

Differentiated graph-modeling studio **[14 min]**

Choose one route.

1. **Core route:** Given the graph of a quadratic cost model, identify intercepts, vertex, and the meaningful domain.
2. **Challenge route:** Fit a sinusoidal model from maximum, minimum, and period; then explain the phase shift.
3. **Extension route:** Choose between a reciprocal model and a square-root model for a congestion or learning-curve dataset. Justify by mechanism and residual pattern.

Lesson summary

Concept	Modeling meaning
Graph	Visual representation of a relation or function; reveals qualitative structure.
Vertical-line test	Checks whether each input has only one output.
Domain and range	Specify realistic inputs and possible outputs.
Intercepts	Describe starting values, thresholds, or break-even points.
Symmetry	Reduces work and reveals structural assumptions.
Transformation	Reuses known graph shapes through shifts, stretches, and reflections.
Sinusoidal model	Represents periodic behaviour through amplitude, period, phase, and midline.
Asymptote	Indicates a limiting value or practical boundary.
Conic	Encodes geometric constraints such as distance, sum of distances, or coverage.

Checkpoint: Exit ticket

Pick one graph feature from today's lesson and explain how it could affect a real decision in a modeling problem. Your answer must include the feature, its context meaning, and one limitation.

Core practice

1. Find the natural domain of $f(x) = \dfrac{\sqrt{9-x^2}}{x-1}$.
2. Determine whether each function is even, odd, or neither: $x^5 - x$, $x^4 + 2x^2$, $x^2 + x$.
3. Describe the transformations from $y = x^2$ to $y = -3(x+2)^2 + 5$.
4. State the amplitude, period, phase shift, and midline of $y = 4\sin(2t - \pi) + 3$.
5. Find the vertical and horizontal asymptotes of $f(x) = \dfrac{2x+1}{x-3}$.

Answer check: core practice

The answers are

$$[-3, 1) \cup (1, 3]; \qquad \text{odd, even, neither;}$$

a shift 2 units left, reflection in the x-axis, vertical stretch by 3, and shift 5 units up; amplitude 4, period π, phase shift $\pi/2$, and midline $y = 3$; vertical asymptote $x = 3$ and horizontal asymptote $y = 2$.

Modeling challenge

A small theme park records the number of visitors entering per hour from 9:00 to 18:00. The data rise slowly in the morning, peak around 14:00, then decline. A student proposes a quadratic model.

1. Define a time variable and a visitor-count function.
2. Explain why a quadratic graph may be reasonable over one day.
3. State the modeling domain and explain why extrapolating beyond it may be misleading.
4. Suggest one graph feature that would help staffing decisions.
5. Write a short graph-based conclusion with one limitation.

Answer check: modeling challenge

A complete response defines time from 9:00 and a visitor-count function, justifies a one-day quadratic as a local rise–peak–fall approximation, restricts the domain to the observed operating day, and identifies the peak time or peak height as a staffing quantity. The conclusion must warn that extrapolation beyond 9:00–18:00 has no support from this model.

Extension practice

1. Prove that the graph of $x^2 + y^2 = 4$ is not the graph of a function of x, but can be split into two functions.
2. A signal is modeled by $S(t) = 2.5\cos(\pi t/6) + 7$. Find its maximum, minimum, period, and value at $t = 3$.

3. Explain how the asymptote in $T(q) = 12 + 40/(100 - q)$ changes if the road capacity increases from 100 to 130 vehicles per minute.
4. Use a circle and its interior to model the coverage points within 50 m of a Wi-Fi router at $(20, 15)$. Write the corresponding inequality.

Answer check: extension practice

The circle splits into $y = \pm\sqrt{4 - x^2}$. For the signal, the maximum is 9.5, the minimum is 4.5, the period is 12, and $S(3) = 7$. Raising road capacity to 130 moves the vertical asymptote from $q = 100$ to $q = 130$ in the correspondingly revised model. The Wi-Fi coverage disk is

$$(x - 20)^2 + (y - 15)^2 \leq 2500.$$

Optional appendix: trigonometric identities

The identities below are useful for algebra and calculus, but they need not all be taught in the two-hour core.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1, \\
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\
\sin(2\theta) &= 2\sin\theta\cos\theta, \\
\cos(2\theta) &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.
\end{aligned}$$

Model audit card

Audit before trusting or communicating a model

Purpose	What question or decision does the model support?	**Data**	Which inputs are observed, derived, or simulated?
Assumptions	Which simplifications may change the answer?	**Units**	Are dimensions and scales consistent?
Verification	Were equations, code, constraints, and limiting cases checked?	**Validation**	Is performance tested against independent evidence?
Uncertainty	How do measurement, parameter, solver, and structural errors affect conclusions?	**Ethics**	Who benefits, who bears risk, and what fairness criterion matters?
Reproducibility	Are data provenance, versions, seeds, and transformations recorded?	**Communication**	Are claims conditional, traceable, and decision-relevant?

Further reading

- Precalculus 2e (OpenStax)

Sources, provenance, and licence

These notes follow the iterative modeling and assessment perspective in the SIAM/COMAP GAIMME report. Competition-specific remarks are a snapshot checked on 20 August 2026 against the official COMAP HiMCM instructions; readers should verify current rules before submission. Unless a source is explicitly identified, classroom datasets and numerical scenarios are synthetic or simulated teaching examples and are not empirical claims about a real institution. Third-party references retain their own terms.

Student response guide and checking criteria

For closed calculations, show the definition, substitution, units, and at least one independent check; the worked examples in this unit are method hints, but you must complete the arithmetic yourself. Open modeling responses are checked on five dimensions: a clear purpose and output; stated data and provenance; defensible assumptions and scope; traceable computational verification and model validation; and alignment among uncertainty, limitations, and the recommendation. Support each dimension with concrete evidence rather than writing only “completed.”