

# Lectures in Mathematical Modeling

## Lecture 7: Decision Theory and Risk

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### Learning goals

By the end of the lesson, readers should be able to:

1. distinguish a **decision alternative** from a **state of nature**;
2. compute and interpret **expected value** for repeated risky decisions;
3. build and solve a simple **decision tree** using the fold-back method;
4. use **Bayes' theorem** to update probabilities after new information;
5. compute the **expected value of perfect information**;
6. apply Laplace, Maximin, Maximax, Hurwicz, and Minimax Regret criteria when probabilities are unknown;
7. explain why the mathematically optimal decision may depend on risk tolerance, repetition, and consequences.

## 1 Why decision theory?

Previous lectures often assumed that the model inputs were known. In real decisions, some inputs are uncertain: the weather may change, market demand may be high or low, a bid may be accepted or rejected, and a medical test may be correct or misleading. **Decision theory** provides a disciplined way to choose actions when outcomes depend on uncertain future states.

A decision model separates three ingredients:

1. **Alternatives**: actions the decision maker can choose.
2. **States of nature**: uncertain conditions not controlled by the decision maker.
3. **Payoffs**: consequences produced by each alternative-state combination.

### Launch: What should the stall sell?

[10 min]

A school fundraising stall can sell either hot coffee or iced tea. Coffee earns more on a cold day; iced tea earns more on a warm day. Tomorrow's weather is uncertain.

In pairs, discuss:

1. What are the alternatives? What are the states of nature?
2. What information would change the decision?
3. Is this a repeated decision or a one-time decision?
4. Would you rather maximize average profit or avoid a very bad outcome?

**Modeling lesson:** Before computing anything, identify who chooses, what is uncertain, and what

consequence matters.

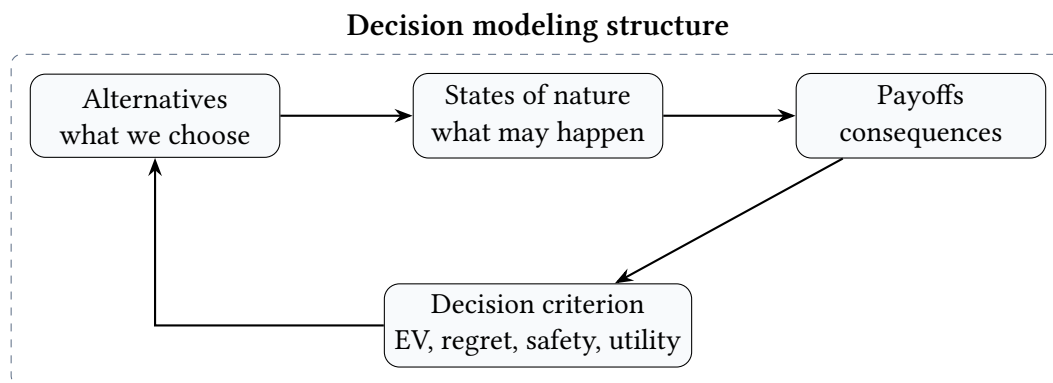


Figure 1: A decision model combines choices, uncertain states, consequences, and a criterion for judging them.

### Repeated decisions versus one-time decisions

**Expected value** is a long-run average. It is most convincing when the same kind of decision is repeated many times, such as an insurance company selling many policies. For a one-time decision with large downside risk, the highest expected value may not be the most acceptable choice. This distinction is not a minor philosophical issue. It determines whether we should use expected value, a risk-averse criterion, or a regret-based criterion.

## 2 Expected value for repeated risky decisions

If a random outcome has payoffs  $w_1, w_2, \dots, w_n$  with probabilities  $p_1, p_2, \dots, p_n$ , where  $\sum p_i = 1$ , its **expected value** is

$$E = \sum_{i=1}^n p_i w_i.$$

The value  $E$  is not necessarily a possible outcome. It is the average payoff expected over many repetitions under the same probability model.

### Worked Example 1: A dice game

A game costs \$1 to play. If two fair dice sum to 7, the player receives \$6 total; otherwise the player receives nothing. The net payoff is therefore \$5 for a sum of 7 and -\$1 otherwise.

There are 36 equally likely dice outcomes and 6 of them sum to 7, so

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}.$$

The expected net payoff is

$$E = 5 \left( \frac{1}{6} \right) + (-1) \left( \frac{5}{6} \right) = 0.$$

The game is **fair** in the expected-value sense. This does not mean every player breaks even; it means that the average net gain approaches zero over many plays.

**Model criticism: Expected value is not a promise**

A student might say, “The expected value is zero, so I will neither win nor lose.” This is wrong. A single play gives either \$5 or −\$1. Expected value describes the centre of the distribution over repeated trials, not the outcome of one trial.

**Worked Example 2: Concession decision with known weather probability**

A concession firm must choose one product for tomorrow’s school sports day.

|          | Cold day | Warm day |
|----------|----------|----------|
| Coffee   | \$4000   | \$1000   |
| Iced tea | \$1500   | \$5000   |

The forecast gives  $P(\text{cold}) = 0.30$ , so  $P(\text{warm}) = 0.70$ .

$$E(\text{coffee}) = 0.30(4000) + 0.70(1000) = 1900.$$

$$E(\text{iced tea}) = 0.30(1500) + 0.70(5000) = 3950.$$

The expected-value decision is to sell iced tea.

The **break-even probability** for cold weather satisfies

$$4000p + 1000(1 - p) = 1500p + 5000(1 - p).$$

Solving gives  $p = \frac{4000}{6500} \approx 0.615$ . Coffee becomes preferable only if the probability of a cold day exceeds about 61.5%.

**Checkpoint: Expected value**

A bid costs \$2000 to prepare. If accepted, it earns a net profit of \$18,000 after subtracting preparation cost. If rejected, the company loses the \$2000 preparation cost.

1. Write the expected value in terms of the acceptance probability  $p$ .
2. Find the break-even value of  $p$ .
3. Explain whether the company should bid if  $p = 0.15$ .

### 3 Decision trees and the fold-back method

A **decision tree** is useful when a decision is followed by one or more uncertain events. We use square nodes for decisions, circular nodes for uncertainty, and terminal values for payoffs.

**Fold-back method**

Solve a decision tree from right to left:

1. At each chance node, compute the expected value of its branches.
2. At each decision node, keep the branch with the best expected value.
3. Replace the solved node by its value and continue moving left.

This method is called **fold-back**, because we simplify the future part of the tree before choosing the earlier action.

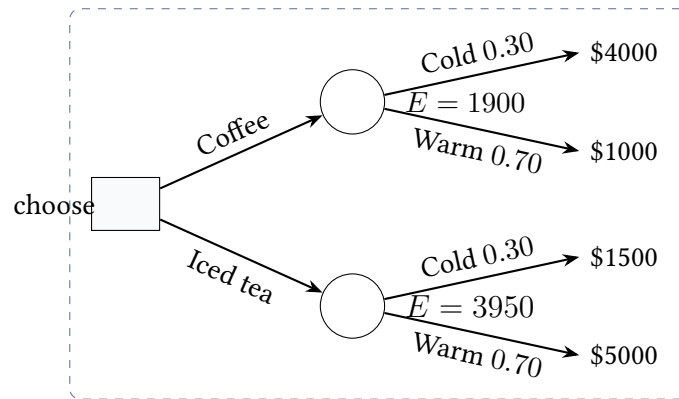


Figure 2: Decision tree for the concession example. Fold-back chooses the branch with the larger expected value.

### Worked Example 3: Plant-size decision

A company may build a large plant, build a small plant, or build nothing. Demand may be high, moderate, or low.

| Alternative | High (0.25) | Moderate (0.40) | Low (0.35) |
|-------------|-------------|-----------------|------------|
| Large plant | 200000      | 100000          | -120000    |
| Small plant | 90000       | 50000           | -20000     |
| No plant    | 0           | 0               | 0          |

$$E(\text{large}) = 0.25(200000) + 0.40(100000) + 0.35(-120000) = 48000.$$

$$E(\text{small}) = 0.25(90000) + 0.40(50000) + 0.35(-20000) = 35500.$$

$$E(\text{none}) = 0.$$

By expected value, the large plant is best. However, the low-demand loss is \$120,000, so the recommendation should mention risk, not only the average.

### Model criticism: Risk profile

Two alternatives can have similar expected values but very different downside risk. A complete decision analysis should report:

1. expected value;
2. best-case and worst-case payoff;
3. probability of loss;
4. whether the decision is repeated or one-time;
5. the decision maker's risk tolerance.

## 4 Information, Bayes' theorem, and EVPI

Information is valuable only if it can change the decision. Decision theory allows us to place an upper bound on the value of information and, when the information is imperfect, to compute whether it is worth buying.

**Perfect information**

The expected value of perfect information is

$$EVPI = E(\text{best action after knowing the state}) - E(\text{best action now}).$$

It is the most we should ever pay for flawless information. Any imperfect forecast or survey must be worth no more than EVPI.

**Worked Example 4: EVPI for the plant-size decision**

Without information, the best expected value is \$48,000 from the large plant.

With perfect information, the company would choose the best action after seeing the true demand state:

| State    | Best action | Payoff   |
|----------|-------------|----------|
| High     | Large plant | \$200000 |
| Moderate | Large plant | \$100000 |
| Low      | No plant    | \$0      |

Therefore,

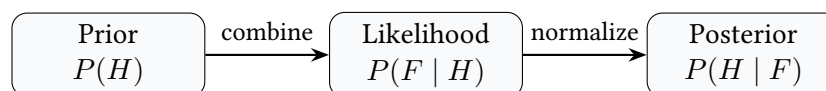
$$E(\text{perfect information}) = 0.25(200000) + 0.40(100000) + 0.35(0) = 90000.$$

$$EVPI = 90000 - 48000 = 42000.$$

No market research report, even a perfect one, is worth more than \$42,000 for this decision.

When information is imperfect, we update probabilities using **Bayes' theorem**. If  $H_1, \dots, H_n$  are possible states and  $F$  is evidence, then

$$P(H_i | F) = \frac{P(F | H_i)P(H_i)}{\sum_j P(F | H_j)P(H_j)}.$$



Bayes update: new evidence changes the probability model before the decision is evaluated.

Figure 3: Bayes' theorem updates beliefs from prior probabilities to posterior probabilities using evidence.

**Worked Example 5: Market research with Bayes update**

For the plant-size problem, suppose a market report is either favorable  $F$  or unfavorable  $U$ . Its historical accuracy is:

| State    | $P(F   \text{state})$ | $P(U   \text{state})$ |
|----------|-----------------------|-----------------------|
| High     | 0.70                  | 0.30                  |
| Moderate | 0.50                  | 0.50                  |
| Low      | 0.20                  | 0.80                  |

First compute

$$P(F) = 0.70(0.25) + 0.50(0.40) + 0.20(0.35) = 0.445.$$

The posterior probabilities after a favorable report are

$$\begin{aligned} P(H | F) &= \frac{0.70(0.25)}{0.445} \approx 0.393, \\ P(M | F) &= \frac{0.50(0.40)}{0.445} \approx 0.449, \\ P(L | F) &= \frac{0.20(0.35)}{0.445} \approx 0.157. \end{aligned}$$

Using the unrounded posterior probabilities, re-evaluating the large plant after a favorable report gives

$$E(\text{large} | F) \approx 104719.$$

Small plant gives about \$54,719, so a favorable report supports the large plant.

For an unfavorable report,  $P(U) = 0.555$  and

$$P(H | U) \approx 0.135, \quad P(M | U) \approx 0.360, \quad P(L | U) \approx 0.505.$$

Using the unrounded posterior probabilities,

$$E(\text{large} | U) \approx 2523, \quad E(\text{small} | U) \approx 20090,$$

so an unfavorable report supports the small plant. The report is useful because it changes the decision in some cases.

### Checkpoint: Value of information

In the market-research example, after a favorable report the decision is large plant, and after an unfavorable report the decision is small plant.

1. Compute the expected value if the company uses the report:  $0.445E_F + 0.555E_U$ .
2. Subtract the best no-report expected value \$48,000 to obtain EVSI.
3. Should the company buy the report if it costs \$10,000? What if it costs \$25,000?

## 5 When probabilities are unknown

Sometimes we cannot assign credible probabilities. In that case, expected value may give a false impression of precision. We use alternative criteria that represent different attitudes toward uncertainty.

### Running Example: Investment payoffs

A student club has funds to invest for one year. Payoffs are in thousand dollars.

| Alternative | Fast growth | Normal growth | Slow growth |
|-------------|-------------|---------------|-------------|
| Stocks      | 10          | 6.5           | -4          |
| Bonds       | 8           | 6             | 1           |
| Savings     | 5           | 5             | 5           |

No reliable probabilities are available for the three economic states.

| Criterion      | Rule                                                               | Decision in example                              |
|----------------|--------------------------------------------------------------------|--------------------------------------------------|
| Laplace        | Treat all states as equally likely and average each row.           | Bonds/Savings tie at 5.0.                        |
| Maximin        | For each row, find the worst payoff; choose the best worst payoff. | Savings.                                         |
| Maximax        | For each row, find the best payoff; choose the best best payoff.   | Stocks.                                          |
| Hurwicz        | Use $\alpha(\text{best}) + (1 - \alpha)(\text{worst})$ .           | Depends on $\alpha$ ; at $\alpha = 0.6$ , Bonds. |
| Minimax regret | Build a regret matrix and minimize the largest regret.             | Bonds.                                           |

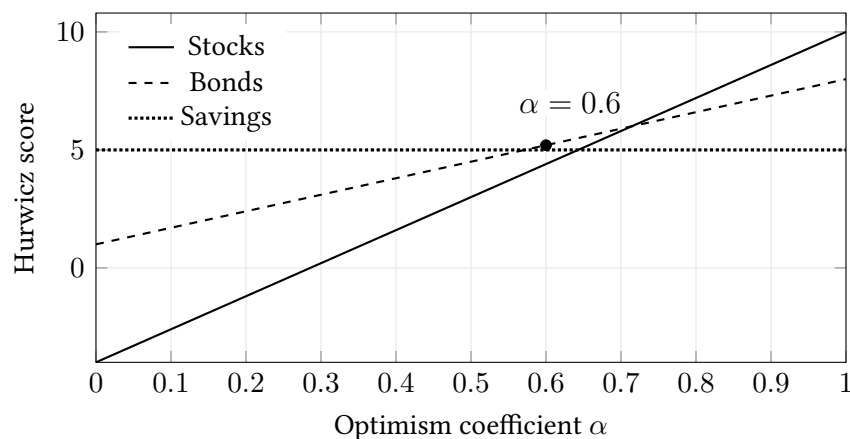


Figure 4: Hurwicz scores change with the optimism coefficient. The chosen alternative depends on the decision maker's attitude toward risk.

#### Worked Example 6: Minimax regret

For each state, subtract each payoff from the best payoff in that column.

|         | Fast          | Normal          | Slow           | Max regret |
|---------|---------------|-----------------|----------------|------------|
| Stocks  | $10 - 10 = 0$ | $6.5 - 6.5 = 0$ | $5 - (-4) = 9$ | 9          |
| Bonds   | $10 - 8 = 2$  | $6.5 - 6 = 0.5$ | $5 - 1 = 4$    | 4          |
| Savings | $10 - 5 = 5$  | $6.5 - 5 = 1.5$ | $5 - 5 = 0$    | 5          |

The smallest maximum regret is 4, so the minimax-regret decision is Bonds. This criterion asks: "If the true state is later revealed, which decision limits how much we will regret not having chosen the best action for that state?"

#### Decision studio

[15 min]

Use the investment example.

- Core route:** Apply Laplace, Maximin, Maximax, and Minimax Regret. Write one sentence explaining each attitude.
- Challenge route:** Find the  $\alpha$  values where Hurwicz changes from Savings to Bonds and from Bonds to Stocks.
- Extension route:** Suppose the club cannot tolerate any possibility of loss. Which criteria respect

this constraint? Which criterion ignores it?

## 6 Integrated mini-case: stocking under uncertain demand

Decision theory becomes most useful when the modeler must connect a business rule, uncertain demand, and a recommendation. The following example is intentionally small enough to compute by hand, but it contains the same structure as larger inventory decisions.

### Worked Example 7: Seasonal T-shirt stocking

A school club can order 10, 20, or 30 festival T-shirts. Each shirt costs \$50 and sells for \$80. Unsold shirts can be cleared for \$20 each. If demand exceeds stock, each unmet demand unit causes a goodwill loss of \$10. Demand is estimated as:

$$P(D = 10) = 0.30, \quad P(D = 20) = 0.40, \quad P(D = 30) = 0.30.$$

For an order quantity  $Q$  and demand  $D$ , the payoff is

$$80 \min(Q, D) + 20 \max(Q - D, 0) - 50Q - 10 \max(D - Q, 0).$$

This gives the payoff matrix:

| Order quantity | $D = 10$ | $D = 20$ | $D = 30$ | Expected value                             |
|----------------|----------|----------|----------|--------------------------------------------|
| $Q = 10$       | 300      | 200      | 100      | $0.30(300) + 0.40(200) + 0.30(100) = 200$  |
| $Q = 20$       | 0        | 600      | 500      | $0.30(0) + 0.40(600) + 0.30(500) = 390$    |
| $Q = 30$       | -300     | 300      | 900      | $0.30(-300) + 0.40(300) + 0.30(900) = 300$ |

By expected value, the best order is  $Q = 20$ .

Different risk criteria tell a more nuanced story:

- Maximin chooses  $Q = 10$ , because its worst payoff is still \$100.
- Maximax chooses  $Q = 30$ , because it can earn \$900 if demand is high.
- Minimax regret chooses  $Q = 20$ : its maximum regret is \$400, smaller than the maximum regret for  $Q = 10$  or  $Q = 30$ .

Thus the same payoff matrix can support different recommendations depending on whether the club prioritizes average profit, downside protection, or regret control.

### Sensitivity check

[8 min]

In the T-shirt example, suppose the probability of high demand  $D = 30$  increases while the probability of low demand  $D = 10$  decreases. Without recalculating every criterion, predict which order quantity becomes more attractive. Then compute the expected values when

$$P(D = 10) = 0.20, \quad P(D = 20) = 0.40, \quad P(D = 30) = 0.40.$$

Explain whether the decision changes.

**Model criticism: Hidden assumptions in payoff matrices**

A payoff matrix looks precise, but each number depends on assumptions: selling price, clearance price, lost goodwill cost, and the demand probabilities. If these inputs are estimated weakly, the report should include sensitivity checks rather than present the final order quantity as certain.

## 7 Writing a defensible decision recommendation

Decision theory does not remove judgment. It makes the judgment explicit. In a modeling report, the recommendation should not simply state the branch with the largest expected value; it should state the assumptions under which the recommendation holds.

**Model criticism: Common mistakes in decision models**

1. Treating guessed probabilities as if they were measured facts.
2. Using expected value for a one-time catastrophic decision without discussing risk.
3. Forgetting to subtract the cost of information from EVSI.
4. Confusing a decision node with a chance node: the modeler chooses at one; nature decides at the other.
5. Reporting a criterion result without explaining the risk attitude represented by the criterion.

**Competition-style conclusion**

For the plant-size case, a concise conclusion might be:

*Using the prior demand probabilities, the large plant has the highest expected value, \$48,000, and is therefore the recommended decision if management is willing to accept a possible \$120,000 loss in the low-demand state. Perfect information would be worth at most \$42,000, so any market research costing more than this should be rejected immediately. Because the large-plant recommendation is sensitive to the low-demand loss and to the estimated probability of high demand, we also recommend reporting a risk profile and considering the small plant if management is strongly risk-averse.*

**Optional extension: Modern audit: mean performance and tail risk**

Expected payoff describes average performance; it does not describe how severe rare losses can be. Write loss as  $L = -W$ . For a confidence level  $\alpha$ , the **value at risk**  $\text{VaR}_\alpha(L)$  is an  $\alpha$ -quantile of loss, while the **conditional value at risk** summarizes the tail beyond that threshold. For a continuous loss distribution,

$$\text{CVaR}_\alpha(L) = E[L \mid L \geq \text{VaR}_\alpha(L)].$$

Thus  $\text{CVaR}_{0.95}$  is the mean loss in the worst 5% of cases. In a finite simulation, estimate it by averaging the worst 5% of simulated losses and report the number of tail observations.

Expected loss answers “What happens on average?”; CVaR answers “How bad are outcomes after we enter the adverse tail?” They are complementary, not interchangeable. A defensible comparison reports expected payoff, probability of loss, a stated tail metric, and sensitivity to uncertain probabilities. CVaR is itself uncertain when extreme observations are scarce, so use confidence intervals or repeated simulation and do not present a noisy tail estimate as a guaranteed safety bound.

**Lesson summary**

1. A decision model separates alternatives, states of nature, probabilities, and payoffs.
2. Expected value is a long-run average and is most appropriate for repeated decisions.
3. Decision trees are solved by fold-back: expected values at chance nodes, best choices at decision nodes.
4. Bayes' theorem updates probabilities after evidence.
5. EVPI is an upper bound on the value of information; EVSI must be compared with the cost of the information.
6. When probabilities are unknown, different criteria represent different attitudes toward risk and uncertainty.

**Checkpoint: Exit ticket****[2 min]**

Complete the sentence:

"Expected value is a useful decision criterion when \_\_\_\_\_, but it may be misleading when \_\_\_\_\_."

## 8 Practice problems

**Core practice**

1. A game costs \$2 to play. A player wins \$12 with probability 0.20 and wins nothing otherwise. Compute the expected net payoff. Is the game fair?
2. A food stall must choose noodles or sandwiches. Profits are:

|            | Rainy ( $p = 0.40$ ) | Sunny ( $p = 0.60$ ) |
|------------|----------------------|----------------------|
| Noodles    | 3200                 | 1600                 |
| Sandwiches | 1800                 | 3600                 |

Compute both expected values and find the break-even probability of rain.

3. For the plant-size problem in Worked Example 3, compute the probability of losing money under each alternative. Does this change how you would present the recommendation?
4. A medical screening test has  $P(+ \mid D) = 0.95$ ,  $P(+ \mid \text{no } D) = 0.08$ , and disease prevalence  $P(D) = 0.02$ . Compute  $P(D \mid +)$ . Explain why a highly sensitive test can still have a moderate positive predictive value.
5. For the investment payoff matrix, apply Hurwicz with  $\alpha = 0.3$  and  $\alpha = 0.8$ . Explain why the decision changes.
6. For the following payoff matrix, construct the regret matrix and apply minimax regret:

|     | $S_1$ | $S_2$ | $S_3$ |
|-----|-------|-------|-------|
| $A$ | 20    | 5     | 8     |
| $B$ | 12    | 12    | 12    |
| $C$ | 4     | 18    | 10    |

**Practice check and hints**

1. The expected net payoff is  $0.20(10) + 0.80(-2) = 0.40$  dollars, so the game is not fair to the organizer in the expected-value sense.
2. Expected profits are 2240 for noodles and 2880 for sandwiches. The break-even rain probability is  $10/17 \approx 0.588$ .
3. Loss probabilities are 0.35, 0.35, and 0 for the large, small, and no-plant choices; loss magnitude must also be reported.
4.  $P(D \mid +) = 0.019/(0.019 + 0.0784) \approx 0.195$ ; low prevalence leaves many false positives relative to true positives.
5. With  $\alpha = 0.3$ , the Hurwicz scores for stocks, bonds, and savings are 0.2, 3.1, and 5, so savings wins. With  $\alpha = 0.8$ , they are 7.2, 6.6, and 5, so stocks win.
6. The regret rows are  $A : (0, 13, 4)$ ,  $B : (8, 6, 0)$ , and  $C : (16, 0, 2)$ ; maximum regrets are 13, 8, and 16, so minimax regret selects  $B$ .

**Modeling challenge**

Choose one of the following and prepare a one-page decision analysis brief containing: alternatives, states of nature, payoff table, probability assumptions if any, chosen criterion, recommendation, sensitivity comment, and one limitation.

1. A school club deciding whether to hold an outdoor fundraising event, indoor event, or cancel.
  2. A student team deciding whether to submit a safe project, a risky innovative project, or a hybrid project to a modeling competition.
  3. A small shop deciding whether to stock 10, 20, or 30 units of a seasonal item.
- Your answer should explain why the chosen criterion is appropriate for the situation.

## A Optional extension: expected utility

Expected value treats each dollar equally. In reality, the pain of losing \$100,000 may be more than twice the pain of losing \$50,000. **Expected utility** replaces payoff  $w$  by a utility function  $u(w)$  and chooses the alternative with largest

$$E[u(W)] = \sum_i p_i u(w_i).$$

A concave utility function represents risk aversion: gaining an extra dollar matters less when the decision maker is already wealthy, while losses near a critical threshold may be unacceptable. This is why a risk-averse manager may choose the small plant even when the large plant has a larger expected monetary value.

## B Optional extension: expected value of sample information

If a report can be favorable or unfavorable, the expected value with sample information is

$$E(\text{with report}) = P(F) \max_a E(a \mid F) + P(U) \max_a E(a \mid U).$$

Then

$$\text{EVSI} = E(\text{with report}) - E(\text{best decision without report}).$$

The report should be purchased only when EVSI exceeds its cost. If the cost is included in terminal payoffs, do not subtract it a second time.

## Model audit card

| Audit before trusting or communicating a model |                                                                                  |                      |                                                                    |
|------------------------------------------------|----------------------------------------------------------------------------------|----------------------|--------------------------------------------------------------------|
| <b>Purpose</b>                                 | What question or decision does the model support?                                | <b>Data</b>          | Which inputs are observed, derived, or simulated?                  |
| <b>Assumptions</b>                             | Which simplifications may change the answer?                                     | <b>Units</b>         | Are dimensions and scales consistent?                              |
| <b>Verification</b>                            | Were equations, code, constraints, and limiting cases checked?                   | <b>Validation</b>    | Is performance tested against independent evidence?                |
| <b>Uncertainty</b>                             | How do measurement, parameter, solver, and structural errors affect conclusions? | <b>Ethics</b>        | Who benefits, who bears risk, and what fairness criterion matters? |
| <b>Reproducibility</b>                         | Are data provenance, versions, seeds, and transformations recorded?              | <b>Communication</b> | Are claims conditional, traceable, and decision-relevant?          |

## Further reading

- Introductory Statistics 2e (OpenStax)
- GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education (SIAM / COMAP)

## Sources, provenance, and licence

These notes follow the iterative modeling and assessment perspective in the SIAM/COMAP GAIMME report. Competition-specific remarks are a snapshot checked on 20 August 2026 against the official COMAP HiMCM instructions; readers should verify current rules before submission. Unless a source is explicitly identified, classroom datasets and numerical scenarios are synthetic or simulated teaching examples and are not empirical claims about a real institution. Third-party references retain their own terms. Original teaching material by Kenneth Sok Kin Cheng is released under CC BY-NC-SA 4.0.

### Student response guide and checking criteria

For closed calculations, show the definition, substitution, units, and at least one independent check; the worked examples in this unit are method hints, but you must complete the arithmetic yourself. Open modeling responses are checked on five dimensions: a clear purpose and output; stated data and provenance; defensible assumptions and scope; traceable computational verification and model validation; and alignment among uncertainty, limitations, and the recommendation. Support each dimension with concrete evidence rather than writing only “completed.”