Lectures in Mathematical Modeling

Lecture 2: The Modeling Process and Scaling

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Learning goals

By the end of the lesson, readers should be able to:

1. turn a vague real-world question into a precise modeling objective;
2. identify variables, units, assumptions, and useful **submodels**;
3. distinguish a proportional relationship from a general linear relationship;
4. test relationships of the form $y \propto f(x)$ by transforming variables;
5. apply **geometric similarity** to derive area, volume, and weight scaling laws;
6. evaluate a model through interpretation, sensitivity, and limitations.

1 From a vague question to a modeling objective

A real-world question rarely arrives in a form that can be solved immediately. The modeler must decide what is being predicted, for whom the answer is useful, and under what conditions the conclusion is intended to hold. This first act of narrowing the question is called **problem formulation**.

Launch: Is the scale model enough? **[10 min]**

A school wants to build a shaded outdoor stage. Before construction, students make a 1:20 scale model and test it under a lamp.
In pairs, discuss:

1. Which measurements from the model can be scaled directly to the full stage?
2. Which effects might not scale in the same way?
3. What decision could the model support?
4. What evidence would make you trust the model?

Modeling lesson: A model can preserve some relationships while distorting others. We must know which quantities scale and why.

A practical modeling workflow

For most competition and classroom problems, the following six questions are more useful than immediately searching for a formula.

1. **Purpose**: What decision, explanation, or prediction is required?
2. **Variables and units:** What must be measured, and in what units?
3. **Assumptions:** What is held constant, neglected, or idealised?

4. **Structure:** Can the problem be separated into smaller submodels?
5. **Evidence:** How will parameters be estimated and results checked?
6. **Communication:** What conclusion is justified, and where might it fail?

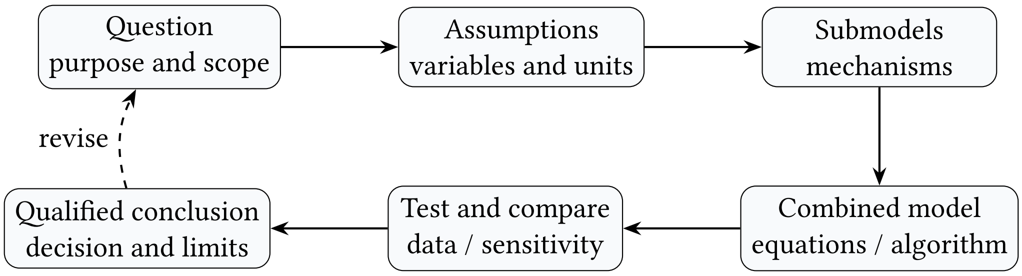

Figure 1: A model is developed through connected decisions, not by choosing a formula in isolation.

Checkpoint: A modelable question

Rewrite the vague question "How safe is this road?" as a more precise modeling question. Your version must identify an output quantity and at least one condition under which the answer applies.

2 Case study: vehicular stopping distance

Consider the question: *How much distance does a car need to stop safely?* A useful first model separates the process into two mechanisms:

$$\text{total stopping distance} = \text{reaction distance} + \text{braking distance}.$$

This is an example of **decomposition**: a complicated system is represented by simpler submodels whose outputs can be combined.

Worked Example 1: Building the stopping-distance model

Let v be the initial speed in metres per second, t_r the driver's reaction time in seconds, and a the magnitude of the braking deceleration in metres per second squared.
Reaction submodel. During the reaction interval, the car is assumed to travel at approximately constant speed:

$$d_r = t_r v.$$

Braking submodel. Once the brakes act, assume constant deceleration. From $0 = v^2 - 2ad_b$,

$$d_b = \frac{v^2}{2a}.$$

Combined model.

$$\boxed{d(v) = t_r v + \frac{v^2}{2a}.}$$

For a baseline driver with $t_r = 1.2$ s and a dry-road deceleration of $a = 6.5$ m/s^2,

$$\boxed{d(v) = 1.2v + 0.0769v^2.}$$

At 50 km/h ($v = 13.89$ m/s), the predicted stopping distance is about 31.5 m. At 100 km/h ($v = 27.78$ m/s), it is about 92.7 m. Doubling speed therefore requires almost three times the distance, not merely twice the distance.

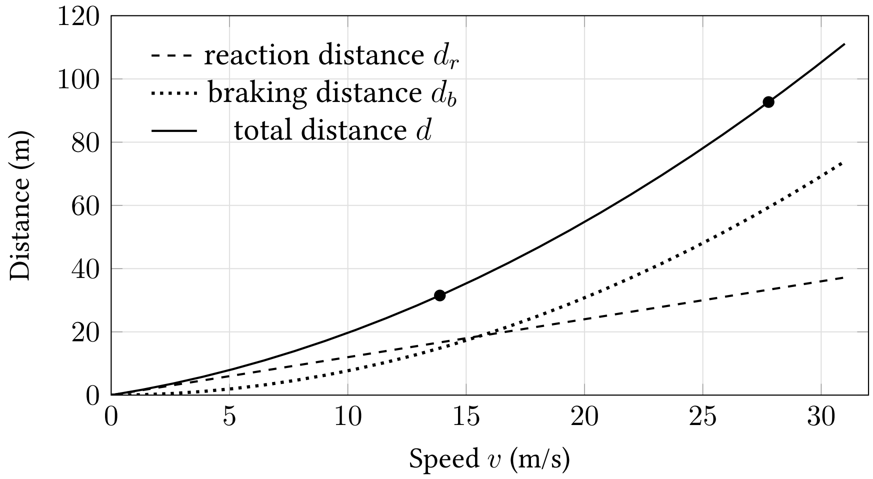

Figure 2: The quadratic braking term becomes increasingly important as speed rises.

Checkpoint: When does braking dominate?

Find the positive speed at which the braking distance first equals the reaction distance. Express the answer in both m/s and km/h. What does this threshold mean for high-speed driving?

A model is not complete when an equation has been obtained. We must ask whether its conclusions are stable under plausible changes in the assumptions.

Worked Example 2: Sensitivity at 80 km/h

At 80 km/h, $v = 22.22$ m/s. Compare four scenarios:

Scenario	t_r (s)	a (m/s^2)	Predicted distance (m)
Baseline: alert driver, dry road	1.2	6.5	64.7
Slower reaction	1.8	6.5	78.0
Wet road	1.2	3.5	97.2
Slower reaction and wet road	1.8	3.5	110.5

For this speed, the conclusion is more sensitive to the available deceleration than to a moderate change in reaction time. This does not mean reaction time is unimportant; it means road and tyre conditions dominate this particular comparison.

Model criticism: What the model leaves out

The model assumes level ground, constant deceleration, immediate full braking after the reaction interval, and no change in direction. It also treats t_r and a as known constants. In reality, they vary with fatigue, distraction, weather, tyre condition, road slope, and braking technology.
A defensible conclusion is therefore conditional: *under the stated reaction time and deceleration assumptions, the required stopping distance increases nonlinearly with speed.*

Refine the model **[8 min]**

Choose one omitted factor—road slope, rain, tyre wear, or driver fatigue.

1. Which parameter or submodel would it affect?
2. Would you represent the factor by a new variable or by changing an existing parameter?

3. What data would be needed to estimate the change?

3 Generalised proportionality and transformed variables

A proportional model need not have the form $y = kx$. More generally,

$$y \propto f(x) \qquad \Longleftrightarrow \qquad y = kf(x).$$

If the relationship is plausible, plotting y against the transformed variable $f(x)$ should produce an approximately straight line through the origin.

Linear is not always proportional

A relationship $y = mx + c$ is **linear** in the school-algebra sense, but it is proportional only when $c = 0$. A taxi fare with a flag-fall charge is linear in distance but not proportional to distance. Likewise, $y = kx^2$ is not a straight line when plotted against x, yet it is proportional to x^2 and becomes a straight line when plotted against the transformed variable x^2.

Worked Example 3: Kepler's third law

Let R be a planet's mean orbital radius measured in astronomical units (AU), and let T be its orbital period in Earth years. Kepler's third law suggests

$$T \propto R^{3/2}.$$

Using Earth, $R = 1$ and $T = 1$, so the proportionality constant is approximately 1:

$$\boxed{T \approx R^{3/2}.}$$

For Jupiter, $R = 5.203$ AU, giving

$$T \approx 5.203^{3/2} \approx 11.87 \text{ years},$$

which is close to the observed 11.86 years.

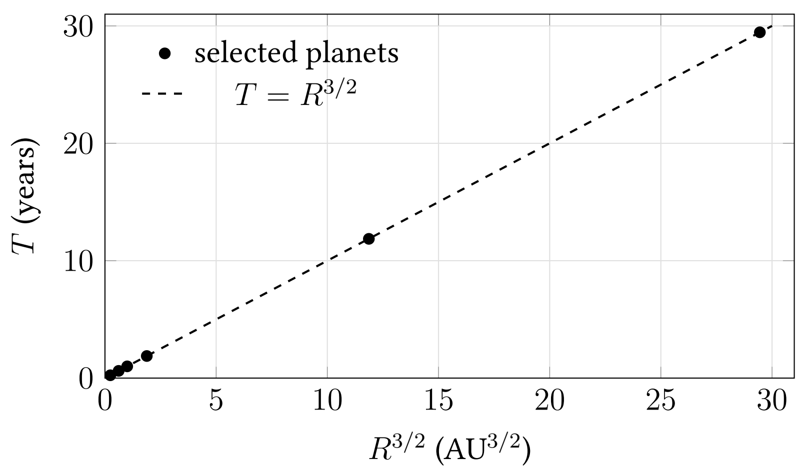

Figure 3: After transforming the horizontal variable, Kepler's relationship is approximately proportional.

Checkpoint: Choose the useful transformation

For each proposed relationship, state what should be plotted on the horizontal axis to test proportionality.

1. $F \propto v^2$
2. $P \propto 1/V$
3. $N \propto e^{rt}$, where r is known

4 Geometric similarity and scaling laws

Two objects are **geometrically similar** if they have the same shape and every corresponding length is multiplied by the same **scale factor** λ.

If all lengths are multiplied by λ, then every area is multiplied by λ^2 and every volume is multiplied by λ^3:

$$\boxed{L \propto \lambda, \qquad S \propto \lambda^2, \qquad V \propto \lambda^3.}$$

For objects of the same material and density, mass and weight also scale as volume:

$$W \propto V \propto L^3.$$

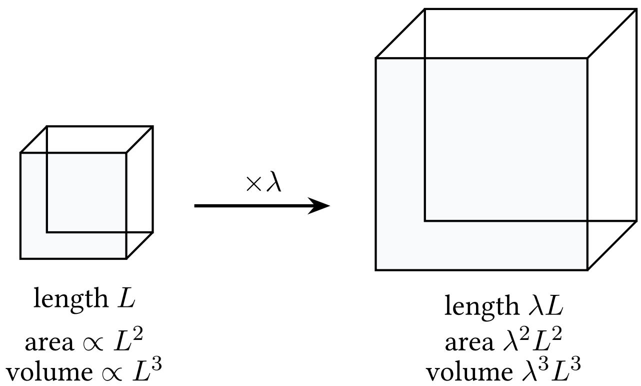

Figure 4: Length, area, and volume do not scale at the same rate.

Scaling table **[6 min]**

Complete the table without using a formula sheet.

Scale factor λ	Length factor	Area factor	Volume factor
2			
3			
$\frac{1}{2}$			

Then explain why a twice-as-tall storage tank does not hold merely twice as much water.

Worked Example 4: A submarine scale model

A 7-m model and a 70-m submarine are geometrically similar, so the full-size vessel has scale factor $\lambda = 10$.

- Corresponding lengths are multiplied by 10.

- Surface areas are multiplied by $10^2 = 100$.
- Volumes and masses are multiplied by $10^3 = 1000$, assuming equal density.

If heat loss is mainly proportional to surface area, the full-size vessel loses about 100 times as much heat under comparable temperature conditions—not 1000 times as much. If storage capacity is proportional to volume, it is about 1000 times as large.

Model criticism: Similarity is an assumption, not an observation

The area and volume laws are exact for mathematically similar shapes. Their application to real objects is only as credible as the similarity assumption. A juvenile animal may not have the same proportions as an adult; a large bridge model may require thicker supports; and a raindrop changes shape as it grows.

5 Model refinement through a scaling model

A useful model is often developed in stages. We begin with the simplest plausible relationship, identify what it misses, and then refine it.

Worked Example 5: Estimating the weight of a fish

Suppose the weight W of a bass must be estimated without a scale.
Model I: length only. If all fish are geometrically similar and have approximately equal density, then

$$W \propto L^3.$$

This model is convenient, but two fish with the same length may have very different body thicknesses.
Model II: length and girth. Approximate the fish as a sequence of similar cross-sections. Cross-sectional area scales with the square of girth G, so

$$\boxed{W \propto LG^2.}$$

Suppose a fish with $L = 40$ cm, $G = 25$ cm weighs 1.25 kg. Then

$$1.25 = k(40)(25)^2 \quad \Longrightarrow \quad k = 0.00005 \text{ kg/cm}^3.$$

A fish with $L = 50$ cm and $G = 30$ cm is predicted to weigh

$$W = 0.00005(50)(30)^2 = 2.25 \text{ kg}.$$

The second model uses one additional measurement to represent body condition more directly.

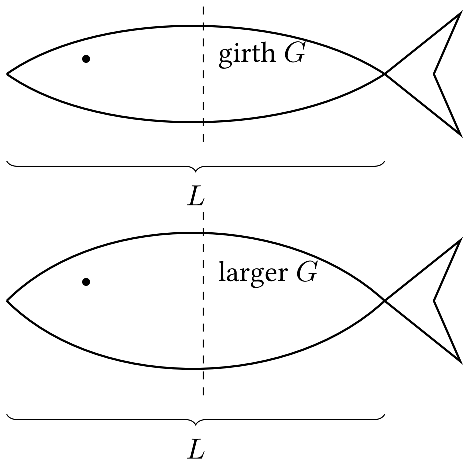

Figure 5: Length alone cannot distinguish fish with different body thicknesses.

Model criticism: Does the refined model solve everything?

The model $W = kLG^2$ still assumes comparable species, density, and body shape. Girth measurements may be inconsistent, and a single calibration fish gives no estimate of prediction error. Refinement improves relevance, but it also introduces an additional measurement and another source of uncertainty.

Checkpoint: Compare two models

For a fish whose length increases by 10% while girth remains unchanged:

1. By what percentage does Model I, $W \propto L^3$, increase its prediction?
2. By what percentage does Model II, $W \propto LG^2$, increase its prediction?
3. Why do the models disagree?

6 A final scaling consequence: strength versus weight

Scaling laws can reveal qualitative limits even when no numerical constant is known. Suppose geometrically similar animals are made of comparable material.

- Muscle strength is approximately proportional to muscle cross-sectional area: strength $\propto L^2$.
- Body weight is proportional to volume: $W \propto L^3$.

Therefore

$$\frac{\text{strength}}{\text{weight}} \propto \frac{L^2}{L^3} = \frac{1}{L} \propto W^{-1/3}.$$

Can a giant ant keep its relative strength? [10 min]

An ant is enlarged so that every length becomes 10 times as large, while shape and material remain unchanged.

1. By what factor does muscle cross-sectional area increase?
2. By what factor does body weight increase?
3. By what factor does strength per unit weight change?
4. State one reason why the geometric-similarity assumption may fail for a real giant animal.

What scaling arguments can and cannot do

A scaling argument often predicts an exponent or qualitative trend without determining the proportionality constant. It is especially useful for comparing sizes and identifying which effects grow faster. It does not replace calibration, validation, or knowledge of the underlying mechanism.

Optional extension: Terminal velocity of similar objects

Suppose geometrically similar falling objects experience quadratic drag. At terminal velocity, weight and drag balance:

$$mg \propto Av_t^2.$$

For constant density, $m \propto L^3$ and frontal area $A \propto L^2$, so

$$L^3 \propto L^2 v_t^2 \quad \Longrightarrow \quad v_t \propto L^{1/2} \propto m^{1/6}.$$

The small exponent means terminal velocity changes relatively slowly with mass. This conclusion depends on quadratic drag, geometric similarity, and comparable density; it is not a universal law for every falling object.

7 Lesson summary

Six ideas to retain

1. A vague question must be narrowed to a purpose, output, and set of conditions.
2. Complicated systems can often be built from interpretable submodels.
3. A proportional relationship $y = kf(x)$ becomes linear through the origin when y is plotted against $f(x)$.
4. In geometrically similar objects, length, area, and volume scale as λ, λ^2, and λ^3.
5. Model refinement should address a specific weakness, not merely add complexity.
6. Every conclusion should be accompanied by assumptions, sensitivity, and a statement of where the model may fail.

Checkpoint: Exit ticket **[2 min]**

Complete both statements:

1. "The most important assumption in today's stopping-distance model is ________________________ because ________________________."
2. "When every length is doubled, ______________ grows faster than ______________ because ____________________."

8 Practice problems

Core practice

1. A driver has reaction time 1.4 s and can decelerate at 5.0 m/s^2.
   (a) Write a stopping-distance model in terms of speed v in m/s.
   (b) Estimate the distance required at 72 km/h.
   (c) Identify one parameter that is uncertain and explain how it could be measured.
2. A delivery fee is 25 dollars plus 4 dollars per kilometre.
   (a) Is fee proportional to distance? Explain.
   (b) Write the linear model and interpret both constants.
3. For each relationship, identify a transformed horizontal variable that would produce a line through the origin if the model were correct:
$$y \propto x^3, \qquad y \propto \sqrt{x}, \qquad y \propto \frac{1}{x^2}.$$
4. Planet X has a mean orbital radius of 4 AU. Use $T \approx R^{3/2}$ to estimate its orbital period. If the observed period is 8.5 years, calculate the absolute relative error using the observed period as the denominator, and suggest one possible reason for the difference.
5. A cube-shaped tank is redesigned so that each side is 30% longer.
   (a) By what factor does its surface area change?
   (b) By what factor does its capacity change?
   (c) Which quantity has the larger percentage increase?
6. A $1{:}25$ model of a monument uses 0.80 kg of material. Estimate the material required for a geometrically similar full-size monument of equal density. State why the estimate may be unrealistic for an actual structure.
7. A fish model has the form $W = kLG^2$. A measured fish has $L = 45$ cm, $G = 28$ cm, and $W = 1.80$ kg.
   (a) Estimate k.
   (b) Predict the weight of a fish with $L = 52$ cm and $G = 31$ cm.
   (c) State two sources of prediction uncertainty.
8. Two geometrically similar animals have weights in the ratio $8{:}1$.
   (a) Find the ratio of their characteristic lengths.
   (b) Find the ratio of their surface areas.
   (c) Compare their surface-area-to-volume ratios.

Practice check and hints

1. $d(v) = 1.4v + v^2/10$; at 72 km/h, $v = 20$ m/s and $d = 68$ m.
2. The fee is $25 + 4d$ dollars and is not proportional because of the nonzero fixed charge.

3. Plot against x^3, $\sqrt{x}$, and $1/x^2$, respectively.
4. The model gives 8 years; relative to 8.5 years, the absolute relative error is about 5.9%.
5. Surface area changes by $1.3^2 = 1.69$ and capacity by $1.3^3 = 2.197$.
6. Volume scaling gives $0.80(25^3) = 12500$ kg; structural design and material thickness need not scale geometrically.
7. $k \approx 5.102 \times 10^{-5}$ kg/cm^3 and the prediction is about 2.55 kg; measurement error and body-shape variation are relevant uncertainties.
8. The length and area ratios are $2{:}1$ and $4{:}1$; the smaller animal has twice the surface-area-to-volume ratio.

Modeling challenge

Choose one question and prepare a one-page modeling brief.

1. How should the dimensions of a school water tank change if student population increases by 50%?
2. Can a small wind-tunnel model predict the aerodynamic force on a full-size vehicle?
3. How can the weight of a fruit be estimated from measurements that do not require a scale?

The emphasis is on the logic of the model. A fully calibrated numerical answer is not required.

One-page modeling brief planner

Purpose and output	What decision or prediction will the model support? What quantity will be reported?
Variables and units	Define the measurable quantities and distinguish inputs from outputs.
Assumptions	State what is treated as constant, neglected, or geometrically similar.
Submodels / scaling	Give at least two relationships and explain why each is plausible.
Calibration	Identify the data needed to estimate any unknown constant.
Validation	Explain what observation or comparison would test the model.
Sensitivity	Name one assumption or parameter that should be varied.
Limitation	State where the conclusion should not be used.

Checkpoint: Submission check

Before submitting, underline the sentence in your brief that gives the main conclusion. Circle every numerical value that has a unit. Put a star beside the assumption that most strongly affects the result.

Model audit card

Audit before trusting or communicating a model

Purpose	What question or decision does the model support?	**Data**	Which inputs are observed, derived, or simulated?
Assumptions	Which simplifications may change the answer?	**Units**	Are dimensions and scales consistent?
Verification	Were equations, code, constraints, and limiting cases checked?	**Validation**	Is performance tested against independent evidence?
Uncertainty	How do measurement, parameter, solver, and structural errors affect conclusions?	**Ethics**	Who benefits, who bears risk, and what fairness criterion matters?
Reproducibility	Are data provenance, versions, seeds, and transformations recorded?	**Communication**	Are claims conditional, traceable, and decision-relevant?

Further reading

- GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education (SIAM / COMAP)
- Guide to the SI: Values, quantities, and dimensions (NIST)
- Planetary fact sheet —ratio to Earth values (NASA NSSDCA)

Sources, provenance, and licence

These notes follow the iterative modeling and assessment perspective in the SIAM/COMAP GAIMME report. Competition-specific remarks are a snapshot checked on 20 August 2026 against the official COMAP HiMCM instructions; readers should verify current rules before submission. Unless a source is explicitly identified, classroom datasets and numerical scenarios are synthetic or simulated teaching examples and are not empirical claims about a real institution. Third-party references retain their own terms.

Student response guide and checking criteria

For closed calculations, show the definition, substitution, units, and at least one independent check; the worked examples in this unit are method hints, but you must complete the arithmetic yourself. Open modeling responses are checked on five dimensions: a clear purpose and output; stated data and provenance; defensible assumptions and scope; traceable computational verification and model validation; and alignment among uncertainty, limitations, and the recommendation. Support each dimension with concrete evidence rather than writing only “completed.”