Macau School Mathematics Competition Archive

澳門校際／學界數學比賽題解資源庫

Independent Solution Companion | 獨立題解讀本

Olympiad Strategy Analysis and Knowledge Map | 奧數思路解析與知識點整理

2019–2025/2026 Solution Companion

2019 至 2025/2026 學年　歷屆初中及高中組獨立題解

This independent teaching companion organises 214 source-verified strategies and worked solutions. Official examination papers remain authoritative and are accessed through the DSEDJ archive.

本獨立教學讀本整理 214 份經來源核對的解題策略與完整解答；正式題目一律以教青局題庫所載原卷為準。

Kenneth Cheng

12 August 2026

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Reader's Guide and Coverage Summary | 閱讀指南與收錄概況

This edition covers eight competition cycles from 2019 to 2025/2026, including both Junior and Senior Divisions. It contains **214 fully structured solutions**, **46 TikZ figure panels**, and **9 editorial notes**. The **16 official bilingual papers are linked rather than reproduced**; open the matching DSEDJ paper before using each solution.
本冊收錄由 2019 至 2025/2026 共八個賽事年度的初中組及高中組，合共 **214 題完整結構化解答**、**46 幅 TikZ 圖形**及 **9 項原題編按**。 **16 份官方中英文原卷只提供連結而不重新封裝**；使用解答前請先開啓相應教青局原卷。

How each solution is organised | 每題結構

Each entry is presented in five layers: an official-paper locator, a concise solution strategy, Olympiad knowledge-point tags, a complete argument, and the final answer. Geometry diagrams are schematic unless coordinates or lengths are explicitly derived in the proof.

每題均依次列出官方原卷定位、思路解析、奧數知識點、完整解答及最終答案。幾何圖如未在證明中建立精確坐標或長度，應視為按題意繪製的講義式示意圖。

Source-verification convention | 原題核對方式

Immediately before each division, this companion provides the official DSEDJ archive link and an attribution date. The examination paper is not embedded, and the official facsimile remains the authoritative record of wording, diagrams, choices, and notation.

每一組別之前均提供教青局官方題庫連結及來源取得日期；本讀本不嵌入原卷，題文、原圖、選項與符號一律以官方影印版為準。

Final clarity review | 最終清晰度檢查

All 214 strategy boxes and all 214 full solutions were checked again for structural completeness and explanatory sufficiency. Short strategy outliers were expanded to state the actual reduction, substitution, extremal idea or equality condition rather than merely naming a technique. Proof-type questions were additionally checked for a visible logical endpoint, and dense algebraic steps were expanded where an unexplained "direct calculation" would otherwise conceal the key mechanism.

全部 214 題的「思路解析」及完整解答已再作一次結構及清晰度檢查。過短的思路已補充至能清楚指出換元、化簡、極值、分類或等號條件；證明題亦檢查是否有明確的邏輯終點。若原稿只以「直接計算」帶過而會遮蔽關鍵機制，則補上必要的中間步驟。

Editorial policy | 編輯原則

Where a scanned paper is ambiguous, internally inconsistent, or apparently missing a condition, the original statement is not silently replaced. The issue is identified in an Editorial Note, and the solution distinguishes what follows from the printed statement from any likely intended version.

若掃描試卷存在歧義、條件矛盾或疑似漏字，本書不會靜默改題，而會以「編按」說明原文能推出的結論及可能的命題原意。

Coverage matrix | 收錄矩陣

Year	**Division**	**Questions**	**Figures**	**Notes**
2019	Junior Division 初中組	13	3	2
2019	Senior Division 高中組	15	3	0
2020	Junior Division 初中組	19	4	2
2020	Senior Division 高中組	16	1	0
2021	Junior Division 初中組	13	5	1
2021	Senior Division 高中組	10	4	0
2022	Junior Division 初中組	16	4	1
2022	Senior Division 高中組	12	4	1
2022/2023	Junior Division 初中組	16	2	0
2022/2023	Senior Division 高中組	12	2	0
2023/2024	Junior Division 初中組	16	5	1
2023/2024	Senior Division 高中組	11	1	0
2024/2025	Junior Division 初中組	16	2	0
2024/2025	Senior Division 高中組	10	3	0
2025/2026	Junior Division 初中組	10	2	1
2025/2026	Senior Division 高中組	9	1	0
Total	**16 division papers**	**214**	**46**	**9**

2019 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

Official Paper Access 官方原卷連結

The examination paper is not reproduced in this public companion. Open the official DSEDJ competition archive, select the matching year and division, and treat that facsimile as authoritative.
本公開讀本不重新封裝原卷。請前往教青局比賽題庫，按學年及組別開啓原卷，並以該原卷為準。

Official archive / 官方題庫： dsedj.gov.mo/cre/science/comp/info.html
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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Prime distribution | 質數分布

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Separate the exceptional intervals containing the prime 2. For all other intervals, every prime is odd; among the five possible odd candidates, at least one is divisible by 3. To determine the minimum, exhibit nine consecutive composite integers.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Residue Classes 同餘類 | Extremal Example 極端例子

Solution 解答

If the interval contains 2, it is either $1, 2, \ldots, 9$ or $2, 3, \ldots, 10$, and each contains exactly four primes. Now suppose the interval does not contain 2. All primes in it are odd. If there are only four odd integers, there are at most four primes. If there are five odd integers, their residues modulo 3 force at least one of them to be divisible by 3. That number is either the prime 3 itself or is composite, so again there are at most four primes.
The upper bound is attained by $1, 2, \ldots, 9$, containing $2, 3, 5, 7$. On the other hand,

$$200, 201, \ldots, 208$$

are all composite. Hence the maximum is 4 and the minimum is 0.

Final Answer 最終答案. $(x, y) = (4, 0)$, so the correct choice is **E**.

Question 2 題目 2 Ratios of fractions | 分數比例

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Convert the numerator and denominator ratios directly into a ratio among the three fractions. Once the ratio of the fractions is known, divide the total sum proportionally.

Olympiad Knowledge Points 奧數知識點

Ratios 比例 **Fractions 分數** **Proportional Allocation 按比例分配**

Editorial Note 編按

As printed, the additional phrase saying that all three fractions are already reduced is incompatible with the two stated integer ratios and the sum $\frac{1}{4}$. The intended ratio calculation nevertheless determines the expected answer below.

Solution 解答

The three fractions are proportional to

$$\frac{4}{1} : \frac{5}{2} : \frac{6}{3} = 4 : \frac{5}{2} : 2 = 8 : 5 : 4.$$

Let them be $8k, 5k, 4k$. Then

$$17k = \frac{1}{4}, \qquad k = \frac{1}{68}.$$

The smallest fraction is therefore

$$4k = \frac{1}{17}.$$

Final Answer 最終答案. $\frac{1}{17}$, which is none of the listed choices.

Question 3 題目 3 Digit sum | 數字和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Write $10^{2019} - 9$ explicitly in decimal form. It consists of a long block of 9's followed by a final digit 1.

Olympiad Knowledge Points 奧數知識點

Decimal Representation 十進制表示 **Digit Sum 數字和**

Solution 解答

We have

$$10^{2019} - 9 = \underbrace{99 \cdots 99}_{2018 \text{ digits}} 1.$$

Multiplication by 10^2 only appends two zeros, so the digit sum is unchanged. Thus

$$S = 2018 \cdot 9 + 1 = 18163.$$

Final Answer 最終答案. 18163, so the correct choice is **C**.

Question 4 題目 4 Integer bounds and ratios | 整數界與比值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

There are only three possible values of $a+b$. For each one, turn the ratio inequality into a very short interval for the integer b.

Olympiad Knowledge Points 奧數知識點

Diophantine Search 丟番圖搜尋 **Sharp Bounds 精確界限**

Solution 解答

Put $n = a + b$, so $n \in \{105, 106, 107\}$ and $a = n - b$. Then

$$0.91 < \frac{n-b}{b} < 0.92$$

is equivalent to

$$\frac{n}{1.92} < b < \frac{n}{1.91}.$$

For $n = 105$ and $n = 106$, the resulting intervals contain no integer. For $n = 107$,

$$55.729\ldots < b < 56.020\ldots,$$

so $b = 56$ and $a = 51$. Therefore

$$21a + 23b = 21(51) + 23(56) = 2359.$$

Final Answer 最終答案. 2359, which is none of the listed choices.

Question 5 題目 5 Fractional parts and estimation | 小數部分與估值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

A complete algebraic simplification is unnecessary. First locate s between two simple rational bounds; this determines its integer part and places $t^2 + t$ between the answer choices.

Olympiad Knowledge Points 奧數知識點

Estimation 估值 **Floor Function 取整函數** **Monotonicity 單調性**

Solution 解答

Since

$$1.9^3 < 7 < 2^3,$$

we have $1.9 < \sqrt[3]{7} < 2$, and hence

$$1 < 3 - \sqrt[3]{7} < 1.1.$$

Therefore

$$\frac{10}{11} < s < 1.$$

Thus $\lfloor s \rfloor = 0$ and $t = s$. Consequently

$$\left(\frac{10}{11}\right)^2 + \frac{10}{11} < t^2 + t < 2.$$

The lower bound is $\frac{210}{121} > \frac{3}{2}$, so

$$\frac{3}{2} < t^2 + t < 2.$$

None of the listed numerical choices equals the answer.

Final Answer 最終答案. The correct choice is **E.**

Question 6 題目 6 Triangle side relation | 三角形邊角關係

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First rewrite the algebraic condition as $b^2 = a^2 + ac$. Then use the sine rule to replace side ratios by trigonometric expressions and reduce the result to a half-angle identity.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Sine Rule 正弦定理** **Half-angle Identities 半角公式**

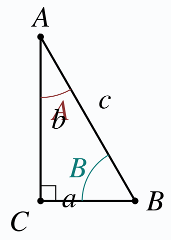

Figure 1. The relation is realised, for example, by a $30°$-$60°$-$90°$ triangle.

Solution 解答

The condition gives

$$a^2 + ac = b^2.$$

By the sine rule,

$$\sin^2 B = \sin^2 A + \sin A \sin C.$$

Equivalently, dividing by $\sin A > 0$ and using $C = \pi - A - B$,

$$\frac{\sin(A+B)}{\sin A} = 1 + 2\cos B.$$

Expanding $\sin(A+B)$ yields

$$\cos A \sin B = \sin A(1 + \cos B).$$

Using

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2}, \qquad 1 + \cos B = 2\cos^2\frac{B}{2},$$

we obtain

$$\cos A \sin\frac{B}{2} = \sin A \cos\frac{B}{2}.$$

Thus $\tan\frac{B}{2} = \tan A$. Since both angles lie in $(0, \pi/2)$,

$$\frac{B}{2} = A.$$

Final Answer 最終答案. $\boxed{B = 2A}$, so the correct choice is **B**.

Question 7 題目 7 Telescoping product 丨連乘積消去

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Factor each term into two adjacent ratios. Almost every numerator and denominator then cancels.

Olympiad Knowledge Points 奧數知識點

Telescoping Product 裂項連乘 **Factorisation 因式分解**

Solution 解答

Since

$$1 - \frac{1}{k^2} = \frac{k-1}{k} \cdot \frac{k+1}{k},$$

we have

$$P = \left(\frac{1}{2}\frac{2}{3}\cdots\frac{2018}{2019}\right)\left(\frac{3}{2}\frac{4}{3}\cdots\frac{2020}{2019}\right).$$

Both products telescope, giving

$$P = \frac{1}{2019} \cdot \frac{2020}{2} = \frac{1010}{2019}.$$

Final Answer 最終答案. $\frac{1010}{2019}$.

Question 8 題目 8 Age ratios 丨年齡比例

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the least common multiple of 5 and 3 to parameterise the four integer ages. The divisibility condition on the average then gives a congruence for the parameter.

Olympiad Knowledge Points 奧數知識點

Linear Diophantine Equations 一次丟番圖 **Congruences 同餘** **Modelling 建模**

Editorial Note 編按

The printed conditions do not determine a unique value of S. The complete family is given below. If the intended question was to find the least possible integer average exceeding 30, the answer is 37.

Solution 解答

Let Ban's age be $15k$. Then the four ages are

$$12k, \quad 15k, \quad 10k, \quad 12k + 1.$$

Hence

$$S = \frac{49k + 1}{4}.$$

For S to be an integer, $k \equiv 3 \pmod 4$. Write $k = 4m + 3$, where $m \geq 0$. Then

$$S = 49m + 37.$$

Every such value satisfies $S > 30$.

Final Answer 最終答案. As written, $\boxed{S = 49m + 37\ (m = 0, 1, 2, \ldots)}$. The least possible value is 37.

Question 9 題目 9 Polynomial coefficients | 多項式係數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Only terms of degree at most 5 are needed. Expand by the binomial theorem and discard all higher-degree terms immediately.

Olympiad Knowledge Points 奧數知識點

Polynomial Expansion 多項式展開 | Degree Truncation 截斷次數

Solution 解答

Put $u = x^2 + x^3$. Then

$$(1 - u)^3 = 1 - 3u + 3u^2 - u^3.$$

Up to degree 5,

$$-3u = -3x^2 - 3x^3,$$

and

$$3u^2 = 3(x^4 + 2x^5 + x^6) = 3x^4 + 6x^5 + \text{higher terms}.$$

The term u^3 begins at degree 6. Therefore

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0 - 3 - 3 + 3 + 6 = 3.$$

Final Answer 最終答案. 3.

Question 10 題目 10 Area and trigonometry ｜面積與三角比

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First observe that $\angle ECD = 20^\circ$, so triangle CED is isosceles. This determines CD from AC. Then compare the two areas using their altitudes to the common line BC.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Area Ratios 面積比 | Sine Rule 正弦定理 | Trigonometric Identities 三角恒等式

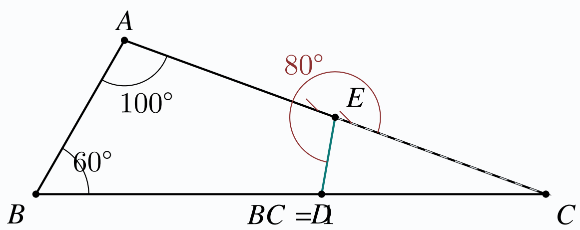

Figure 2. The midpoint condition and the angles make $\triangle CED$ isosceles.

Solution 解答

Since $\angle C = 20^\circ$ and D lies on BC,

$$\angle ECD = 20^\circ.$$

Together with $\angle CED = 80^\circ$, this gives $\angle CDE = 80^\circ$, so

$$CD = CE = \frac{AC}{2}.$$

Let h be the altitude from A to BC. Since E is the midpoint of AC, its altitude to BC is $h/2$. Hence

$$S_{ABC} = \frac{h}{2}, \qquad S_{CDE} = \frac{1}{2} \cdot CD \cdot \frac{h}{2} = \frac{AC \cdot h}{8}.$$

Therefore

$$S_{ABC} + 2S_{CDE} = \frac{h(2 + AC)}{4}.$$

By the sine rule,

$$AC = \frac{\sin 60^\circ}{\sin 100^\circ} = \frac{\sin 60^\circ}{\sin 80^\circ}, \qquad h = AC \sin 20^\circ.$$

Thus

$$S_{ABC} + 2S_{CDE} = \frac{\sin 60^\circ \sin 20^\circ}{4 \sin 80^\circ}\left(2 + \frac{\sin 60^\circ}{\sin 80^\circ}\right).$$

Using

$$\sin 20^\circ (2 \sin 80^\circ + \sin 60^\circ) = \sin^2 80^\circ,$$

the expression simplifies to

$$\frac{\sin 60^\circ}{4} = \frac{\sqrt{3}}{8}.$$

Final Answer 最終答案. $\frac{\sqrt{3}}{8}$.

Question 11 題目 11 Floor-sum pairing | 取整和配對

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Pair the kth term with the $(503 - k)$th term. Coprimality ensures that neither fraction is an integer, so each pair has a constant sum.

Olympiad Knowledge Points 奧數知識點

Floor Sums 取整和 | **Complementary Pairing 互補配對** | **Coprimality 互素**

Solution 解答

Since $\gcd(305, 503) = 1$, the number $305k/503$ is not an integer for $1 \le k \le 502$. Therefore

$$\left\lfloor \frac{305k}{503} \right\rfloor + \left\lfloor \frac{305(503-k)}{503} \right\rfloor = \lfloor x \rfloor + \lfloor 305 - x \rfloor = 304.$$

There are 251 pairs $\{k, 503 - k\}$. Hence

$$S = 251 \cdot 304 = 76304.$$

Final Answer 最終答案. 76304.

Question 12 題目 12 Tiling invariant 丨鋪砌不變量

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Area alone does not decide the problem. Assign the periodic complex weights $1, i, -1, -i$ along both coordinate directions. Every 1×4 tile has total weight 0, but the whole board does not.

Olympiad Knowledge Points 奧數知識點

Combinatorial Geometry 組合幾何 **Colouring Invariant 染色不變量** **Roots of Unity 單位根**

1	i	-1	$-i$
i	-1	$-i$	1
-1	$-i$	1	i
$-i$	1	i	-1

Figure 3. A 4×4 fundamental block of the periodic complex weighting.

Solution 解答

Number rows and columns from 0 to 9, and give cell (r, c) the weight

$$i^{r+c}.$$

Any horizontal 1×4 rectangle has weight

$$i^m(1 + i + i^2 + i^3) = 0,$$

and the same is true for any vertical rectangle. Thus every proposed tiling would have total weight 0.
However, the total weight of the board is

$$\left(\sum_{r=0}^{9} i^r\right)\left(\sum_{c=0}^{9} i^c\right) = (1 + i)^2 = 2i \neq 0.$$

This contradiction proves that the tiling is impossible.

Final Answer 最終答案. No, such a tiling is impossible.

Question 13 題目 13 Rationality identity | 有理性恒等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Introduce the three differences $x = a - b$, $y = b - c$, $z = c - a$. Their sum is zero, which turns the numerator obtained after combining the fractions into a perfect square.

Olympiad Knowledge Points 奧數知識點

Algebraic Identities 代數恒等式 **Rationality 有理性** **Symmetric Expressions 對稱式**

Solution 解答

Put

$$x = a - b, \qquad y = b - c, \qquad z = c - a.$$

Then $x + y + z = 0$ and $xyz \neq 0$. We have

$$S^2 = \frac{x^2y^2 + y^2z^2 + z^2x^2}{x^2y^2z^2}.$$

Since $x + y + z = 0$,

$$x^2y^2 + y^2z^2 + z^2x^2 = (xy + yz + zx)^2.$$

Therefore

$$S = \frac{|xy + yz + zx|}{|xyz|}.$$

All quantities on the right are rational, so S is rational.

Final Answer 最終答案. Yes. In fact $S = \dfrac{|xy + yz + zx|}{|xyz|} \in \mathbb{Q}$.

2019 | Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Recurring decimals | 循環小數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use $9990 = 10 \cdot 999$ and the standard identity $\frac{109}{999} = 0.\overline{109}$.

Olympiad Knowledge Points 奧數知識點

Decimal Expansion 小數展開 | Recurring Decimals 循環小數

Solution 解答

We have

$$\frac{109}{9990} = \frac{1}{10} \cdot \frac{109}{999} = \frac{1}{10} \cdot 0.\overline{109} = 0.0\overline{109}.$$

Thus the repeating block is 109, beginning after the first decimal zero.

Final Answer 最終答案. $0.0\overline{109}$, so the correct choice is **B**.

Question 2 題目 2 Egyptian fractions | 埃及分數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

First order the denominators $x_1 \le x_2 \le x_3$. The smallest denominator can only be 2 or 3, after which the remaining possibilities are forced. Finally count all permutations.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Egyptian Fractions 埃及分數 | Ordered Counting 有序計數

Solution 解答

Assume first that $x_1 \le x_2 \le x_3$. Then

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \le \frac{3}{x_1},$$

so $x_1 \le 3$. Also $x_1 \ne 1$.
If $x_1 = 3$, then all three reciprocals are at most $1/3$ and their sum is 1, so

$$(x_1, x_2, x_3) = (3, 3, 3).$$

If $x_1 = 2$, then

$$\frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{2}.$$

Since $x_2 \le x_3$, we have $x_2 \le 4$. The cases $x_2 = 3, 4$ give

$$(2, 3, 6), \qquad (2, 4, 4).$$

Thus the unordered types are

$$(3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6).$$

Their numbers of ordered permutations are $1, 3, 6$, respectively. Hence

$$N = 1 + 3 + 6 = 10.$$

Final Answer 最終答案. $N = 10$, which is none of the listed choices.

Question 3 題目 3 Reduced fractions and Euler's function | 既約分數與歐拉函數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Pair each numerator a coprime to 2019 with $2019 - a$. Each pair of fractions sums to 1. The number of admissible numerators is $\varphi(2019)$.

Olympiad Knowledge Points 奧數知識點

Euler Totient 歐拉函數 | **Complementary Pairing 互補配對** | **Coprime Residues 互素剩餘類**

Solution 解答

Since

$$2019 = 3 \cdot 673$$

and 673 is prime,

$$\varphi(2019) = 2019\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{673}\right) = 1344.$$

If $\gcd(a, 2019) = 1$, then $\gcd(2019 - a, 2019) = 1$, and

$$\frac{a}{2019} + \frac{2019 - a}{2019} = 1.$$

The 1344 fractions form 672 complementary pairs. Therefore

$$S = 672.$$

Final Answer 最終答案. 672, which is none of the listed choices.

Question 4 題目 4 Handshake counting | 握手計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Separate the handshakes into man-man and man-woman handshakes. This avoids double-counting and makes the restrictions transparent.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Double Counting 雙重計數** **Graph Edges 圖的邊數**

Solution 解答

Among the n men, every pair shakes hands, contributing

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Each man also shakes hands with the $n-1$ women other than his wife, contributing

$$n(n-1).$$

There are no woman-woman handshakes. Hence

$$S = \frac{n(n-1)}{2} + n(n-1) = \frac{3n(n-1)}{2}.$$

Final Answer 最終答案. $S = \dfrac{3n(n-1)}{2}$, which is none of the listed choices.

Question 5 題目 5 Ellipse and focal property | 橢圓與焦點性質

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Recognise that $A = (0, 4)$ is a focus of the ellipse. Replace $|AM|$ using the constant focal sum, then apply the reverse triangle inequality to the other focus and B.

Olympiad Knowledge Points 奧數知識點

Analytic Geometry 解析幾何 **Ellipse Focus 橢圓焦點** **Reverse Triangle Inequality 反三角不等式**

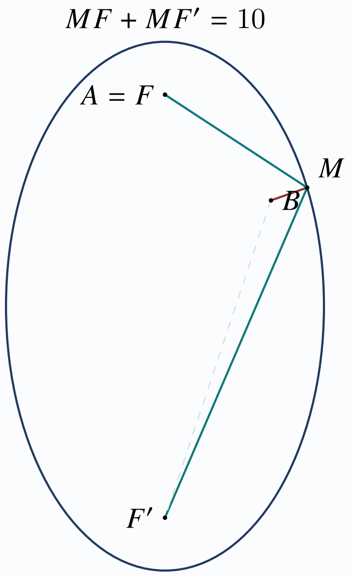

Figure 4. The equality point lies on the line through F' and B, extended beyond B.

Solution 解答

The ellipse has semi-major axis 5 and focal distance

$$\sqrt{25 - 9} = 4.$$

Hence its foci are

$$F = (0, 4) = A, \qquad F' = (0, -4),$$

and for every $M \in E$,

$$|MF| + |MF'| = 10.$$

Therefore

$$|AM| + |MB| = 10 - |MF'| + |MB|.$$

By the triangle inequality,

$$|MF'| \le |MB| + |BF'|,$$

so

$$|AM| + |MB| \ge 10 - |BF'|.$$

Now

$$|BF'| = \sqrt{(2-0)^2 + (2+4)^2} = 2\sqrt{10}.$$

Equality occurs when M, B, F' are collinear with B between M and F', and that ray meets the ellipse. Thus the minimum is attained.

Final Answer 最終答案. $\boxed{10 - 2\sqrt{10}}$.

Question 6 題目 6 Polynomial functional equation | 多項式函數方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

View the identity as a semiconjugacy between two quadratic maps. Symmetry about the critical point forces the polynomial to be even or odd. The odd case contradicts growth along an escaping orbit; the even case halves the degree and can be repeated indefinitely.

Olympiad Knowledge Points 奧數知識點

Polynomials 多項式 | Functional Equations 函數方程 | Symmetry 對稱性 | Degree Descent 次數遞降

Solution 解答

Constant solutions satisfy
$$P^2 - 1 = 4P,$$
so
$$P \equiv 2 + \sqrt{5} \quad \text{or} \quad P \equiv 2 - \sqrt{5}.$$
We prove that there is no nonconstant solution.
Set
$$F(x) = x^2 - 4x + 1 = (x-2)^2 - 3, \qquad G(y) = \frac{y^2 - 1}{4}.$$
The equation is $P(F(x)) = G(P(x))$. Since
$$F(2+t) = F(2-t),$$
we have
$$P(2+t)^2 = P(2-t)^2.$$
Thus the product
$$\big(P(2+t) - P(2-t)\big)\big(P(2+t) + P(2-t)\big)$$
is identically zero. Hence P is either even or odd about 2.
Suppose first that P is odd about 2. Then $P(2) = 0$. Define
$$x_0 = 2, \qquad x_{n+1} = F(x_n).$$
The orbit begins
$$2, -3, 22, 397, \ldots$$

and tends to $+\infty$. On the other hand,

$$P(x_{n+1}) = G(P(x_n)),$$

so $P(x_n) = G^n(0)$. Since G maps the interval $[-1/4, 0]$ into itself, this sequence is bounded. A nonconstant polynomial cannot remain bounded along a sequence tending to $+\infty$, a contradiction.
Therefore P is even about 2, so for some polynomial Q,

$$P(x) = Q((x-2)^2).$$

Putting $u = (x-2)^2$ in the original identity gives

$$Q(u)^2 - 1 = 4Q((u-5)^2). \tag{1}$$

Let $H(u) = (u-5)^2$. As before, $H(5+t) = H(5-t)$, so Q is either even or odd about 5.
If Q is odd about 5, then $Q(5) = 0$. The orbit

$$5, 0, 25, 400, \ldots$$

under H tends to $+\infty$, while equation (1) again makes the corresponding Q-values equal to the bounded orbit of 0 under G. This is impossible.
Hence Q is even about 5, so

$$Q(u) = Q_1((u-5)^2) = Q_1(H(u))$$

for another polynomial Q_1. Substitution into (1) shows that Q_1 satisfies the same equation (1), while

$$\deg Q_1 = \frac{1}{2}\deg Q.$$

Repeating the argument would force $\deg Q$ to be divisible by every power of 2. Therefore $\deg Q = 0$, contradicting the assumption that P is nonconstant.
Thus only the two constant solutions exist.

Final Answer 最終答案. $\boxed{P(x) \equiv 2+\sqrt{5} \text{ or } P(x) \equiv 2-\sqrt{5}}$.

Question 7 題目 7 Maximum under a cubic constraint 丨三次約束下的最大值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Relate $(x+y)^3$ to x^3+y^3 using convexity or the standard inequality $(x+y)^3 \le 4(x^3+y^3)$.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Convexity 凸性 | Equality Cases 等號條件

Solution 解答

For nonnegative x, y,

$$(x+y)^3 \le 4(x^3+y^3).$$

Therefore

$$(x+y)^3 \le 8, \qquad x+y \le 2.$$

Equality is attained at $x = y = 1$, for which $x^3 + y^3 = 2$.

Final Answer 最終答案. $\boxed{2}$.

Question 8 題目 8 Exponential Diophantine equation | 指數丟番圖方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Compare prime exponents on both sides. The exponent vectors of x and y must be proportional by a power of 2, reducing the problem to a small growth comparison.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Prime Valuations 質因數指數 | Exponential Growth 指數增長

Solution 解答

Since $512^y = 2^{9y}$, the equation is

$$x^{2^x} = y^{2^{9y}}.$$

For every prime p, let $v_p(x) = a_p$ and $v_p(y) = b_p$. Then

$$a_p 2^x = b_p 2^{9y}.$$

Thus x and y have the same prime divisors.
If $x \le 9y$, put $m = 9y - x \ge 0$. Then

$$a_p = 2^m b_p$$

for every p, and hence

$$x = y^{2^m}. \tag{1}$$

If $y = 1$, equation (1) gives $x = 1$, which is a solution.

Now let $y \geq 2$. The case $m = 0$ would give $x = y$ and $x = 9y$, impossible. Hence $m \geq 1$, so $x \geq y^2$. Since also $x \leq 9y$, we get $y \leq 9$. Checking $2 \leq y \leq 9$ in

$$y^{2^m} = 9y - m$$

shows that the only possibility is

$$y = 2, \qquad m = 2, \qquad x = 16.$$

Indeed,

$$16^{2^{16}} = (2^4)^{2^{16}} = 2^{2^{18}} = 2^{512^2}.$$

If $x > 9y$, the same valuation argument gives

$$y = x^{2^{x-9y}} \geq x^2,$$

which contradicts $x > 9y \geq 9x^2$ for positive integers. Therefore there are no further solutions.

Final Answer 最終答案. $\boxed{(x, y) = (1, 1), (16, 2)}$.

Question 9 題目 9 Extremal residues modulo 3 | 模 3 極值問題

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Only the residues modulo 3 matter. Classify which residue triples make $ab + bc + ca$ divisible by 3, then maximise a set avoiding all such triples.

Olympiad Knowledge Points 奧數知識點

Extremal Combinatorics 極值組合 **Modular Arithmetic 模算術**
Pigeonhole Principle 鴿巢原理

Solution 解答

Reduce all integers modulo 3. A triple is good if all three residues are equal, because

$$r^2 + r^2 + r^2 = 3r^2 \equiv 0 \pmod 3.$$

A triple containing two residues 0 is also good. The remaining residue patterns

$$(0, 1, 1), (0, 2, 2), (0, 1, 2), (1, 1, 2), (1, 2, 2)$$

do not give 0 modulo 3.

A set with no good triple can therefore contain at most two numbers of residue 1 and at most two of residue 2. It can contain at most one number of residue 0 if it contains any other number. Thus its size is at most

$$1 + 2 + 2 = 5.$$

This bound is attainable, for example by residues

$$0, 1, 1, 2, 2.$$

Hence every set of six distinct integers contains a required triple, while five do not suffice.

Final Answer 最終答案. $\boxed{n = 6}$.

Question 10 題目 10 Nonlinear recurrence and hidden linearity | 非線性遞推與隱藏線性

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Square the radical relation to obtain an invariant involving two consecutive terms. Reuse the same invariant one step later to identify the next radical exactly.

Olympiad Knowledge Points 奧數知識點

Sequences 數列 **Invariant 不變式** **Linearisation 線性化**

Solution 解答

From the recurrence,

$$u_{n+1} - 2u_n = \sqrt{3u_n^2 + 1}.$$

Squaring gives

$$u_{n+1}^2 - 4u_n u_{n+1} + u_n^2 = 1. \tag{1}$$

Now

$$(2u_{n+1} - u_n)^2 = 4u_{n+1}^2 - 4u_n u_{n+1} + u_n^2 = 3u_{n+1}^2 + 1$$

by (1). Since all terms are positive,

$$\sqrt{3u_{n+1}^2 + 1} = 2u_{n+1} - u_n.$$

Therefore

$$u_{n+2} = 2u_{n+1} + 2u_{n+1} - u_n = 4u_{n+1} - u_n.$$

Hence

$$S_n = 0.$$

Final Answer 最終答案. $\boxed{S_n = 0}$ for every $n \geq 1$.

Question 11 題目 11 Quadratic conjugates and recurrence | 二次共軛與遞推

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The two bases are conjugate roots of $t^2-4t+1=0$. Their power sum is therefore an integer sequence satisfying a second-order recurrence, which can be studied modulo 10.

Olympiad Knowledge Points 奧數知識點

Algebraic Conjugates 代數共軛 **Linear Recurrence 線性遞推**
Modular Periodicity 模週期

Solution 解答

Let
$$V_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n.$$
Since $2 \pm \sqrt{3}$ are roots of $t^2 - 4t + 1 = 0$,
$$V_n = 4V_{n-1} - V_{n-2}, \qquad V_0 = 2, \qquad V_1 = 4.$$
Modulo 10, the sequence begins
$$2, 4, 4, 2, 4, 4, \ldots$$
with period 3. Since $2019 \equiv 0 \pmod 3$,
$$V_{2019} \equiv 2 \pmod{10}.$$

Final Answer 最終答案. The units digit is $\boxed{2}$.

Question 12 題目 12 Perfect squares and powers of two | 完全平方與二次冪

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

First eliminate $n < 8$ by parity of prime exponents. For $n \geq 8$, factor out 2^8 and reduce the problem to two powers of 2 differing by 2.

Olympiad Knowledge Points 奧數知識點

Diophantine Equations 丟番圖方程 **Powers of Two 二次冪**
Difference of Squares 平方差

Solution 解答

Suppose
$$2^8 + 2^n = m^2.$$
If $n < 8$, then
$$m^2 = 2^n(1 + 2^{8-n}),$$
where the second factor is odd. The exponent of 2 forces n to be even, but direct substitution $n = 2, 4, 6$ gives no square.
Now let $n \geq 8$ and write $k = n - 8$. Then
$$m^2 = 2^8(1 + 2^k),$$
so $m = 16q$ and
$$q^2 = 1 + 2^k.$$
Thus
$$(q-1)(q+1) = 2^k.$$
Both factors are powers of 2 and differ by 2, so necessarily
$$q - 1 = 2, \qquad q + 1 = 4.$$
Hence $q = 3$, $k = 3$, and
$$n = 11.$$
Indeed, $2^8 + 2^{11} = 2304 = 48^2$.

Final Answer 最終答案. $\boxed{n = 11}$.

Question 13 題目 13 *A*-excircle inequality | *A*-旁切圓不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Express the A-exradius and the side BC using the circumradius and half-angle formulas. The inequality then reduces to a one-line product-to-sum estimate.

Olympiad Knowledge Points 奧數知識點

Euclidean Geometry 歐氏幾何 **Excircle 旁切圓** **Half-angle Formula 半角公式** **Product-to-sum 積化和差**

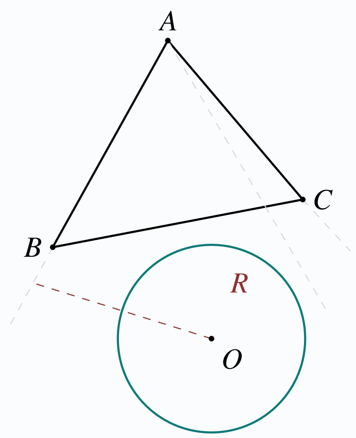

Figure 5. The A-excircle is tangent to BC and to the extensions of AB and AC.

Solution 解答

Let $\mathcal{R}$ be the circumradius of triangle ABC. The standard half-angle formulas give

$$R = 4\mathcal{R}\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

while

$$BC = 2\mathcal{R}\sin A = 4\mathcal{R}\sin\frac{A}{2}\cos\frac{A}{2}.$$

Therefore

$$\frac{2R}{BC} = \frac{2\cos(B/2)\cos(C/2)}{\cos(A/2)}.$$

Now

$$2\cos\frac{B}{2}\cos\frac{C}{2} = \cos\frac{B+C}{2} + \cos\frac{B-C}{2} = \sin\frac{A}{2} + \cos\frac{B-C}{2} \le 1 + \sin\frac{A}{2}.$$

Dividing by $\cos(A/2) > 0$ proves the claim.

Final Answer 最終答案. The stated inequality holds, with equality when $B = C$.

Question 14 題目 14 Unit-distance incidence bound | 單位距離關聯上界

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Associate each blue point with the unordered pair of red points at distance 1 from it. A pair of red points has at most two common unit-distance points. This gives an incidence bound, and a generic geometric construction attains it.

Olympiad Knowledge Points 奧數知識點

Extremal Geometry 極值幾何 **Double Counting 雙重計數**
Geometric Construction 幾何構造

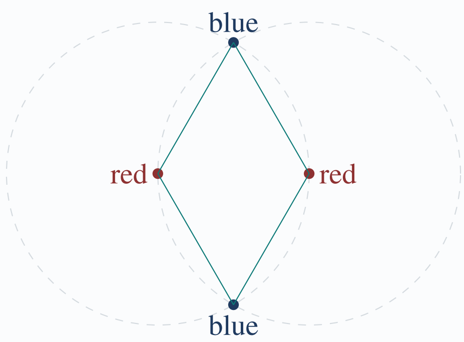

Figure 6. One unordered red pair can generate at most two blue centres.

Solution 解答

Let R and B be the numbers of red and blue points. Each blue point determines the unordered pair of the two red points on its unit circle.
For a fixed pair of red points, there are at most two points at distance 1 from both of them, because they are intersections of two unit circles. Hence

$$B \le 2\binom{R}{2} = R(R-1).$$

Since $B + R = 2019$,

$$2019 \le R^2.$$

Thus $R \ge 45$, and so

$$B \le 2019 - 45 = 1974.$$

It remains to show attainability. Choose 45 red points in a sufficiently small disk and in generic position, so every red pair is less than distance 2 apart, no three red points lie on a unit circle, and the unit-circle intersection points arising from distinct red pairs are all distinct. Every red pair then gives two candidate blue points, each incident with exactly those two red points. There are

$$2\binom{45}{2} = 1980$$

such candidates. Select any 1974 of them as blue points. The required condition holds, and the total number of points is $45 + 1974 = 2019$.

Final Answer 最終答案. $\boxed{1974}$.

Question 15 題目 15 Equal-sum partition | 等和分組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Divide the numbers into eleven consecutive blocks of length 11. Form each group by taking one entry from every block, with the positions cyclically shifted. Each group then has the same quotient sum and the same residue sum.

Olympiad Knowledge Points 奧數知識點

Constructive Combinatorics 構造組合 **Latin-square Pattern 拉丁方模式** **Modular Arrangement 模排列**

Solution 解答

The total sum is

$$1 + 2 + \cdots + 121 = \frac{121 \cdot 122}{2} = 7381,$$

so each group must have sum

$$\frac{7381}{11} = 671.$$

One valid construction is:

1	1	13	25	37	49	61	73	85	97	109	121
2	2	14	26	38	50	62	74	86	98	110	111
3	3	15	27	39	51	63	75	87	99	100	112
4	4	16	28	40	52	64	76	88	89	101	113
5	5	17	29	41	53	65	77	78	90	102	114
6	6	18	30	42	54	66	67	79	91	103	115
7	7	19	31	43	55	56	68	80	92	104	116
8	8	20	32	44	45	57	69	81	93	105	117
9	9	21	33	34	46	58	70	82	94	106	118
10	10	22	23	35	47	59	71	83	95	107	119
11	11	12	24	36	48	60	72	84	96	108	120

Each row has sum 671, and every integer from 1 to 121 appears exactly once.

Final Answer 最終答案. Yes. The table above gives an explicit partition into eleven groups, each of sum 671.

2020 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Addition chains | 加法鏈

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Construct an addition chain. A Fibonacci-type beginning produces many useful values efficiently; the final few terms are then adjusted so that two earlier terms add exactly to 2020.

Olympiad Knowledge Points 奧數知識點

Combinatorial Construction 組合構造 **Addition Chains 加法鏈** **Backward Design 逆向設計**

Solution 解答

One valid sequence is

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a_k	1	2	3	5	8	13	21	34	55	89	144	233	377	382	437	819	1201	2020.

Indeed, the first thirteen terms follow the Fibonacci rule. The remaining terms satisfy

$$382 = 5 + 377, \quad 437 = 55 + 382, \quad 819 = 382 + 437,$$

$$1201 = 382 + 819, \qquad 2020 = 819 + 1201.$$

Every displayed sum uses two distinct earlier terms, as required.

Final Answer 最終答案. The sequence displayed above is a valid construction.

Question 2 題目 2 Consecutive squares | 連續整數平方

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Solution Strategy 思路解析

Search among the smallest consecutive integers and then verify the required partition directly. Since the question only asks for an example, a correct explicit identity is sufficient.

Olympiad Knowledge Points 奧數知識點

Constructive Example 構造例子 **Pythagorean-type Identity 平方和恒等式**

Solution 解答

Take the five consecutive positive integers

$$2, 3, 4, 5, 6.$$

Then

$$3^2 + 6^2 = 9 + 36 = 45,$$

while

$$2^2 + 4^2 + 5^2 = 4 + 16 + 25 = 45.$$

Thus the squares of 3 and 6 have the same sum as the squares of the remaining three integers.

Final Answer 最終答案. $2, 3, 4, 5, 6$, with $3^2 + 6^2 = 2^2 + 4^2 + 5^2$.

Question 3 題目 3 Cubic residues | 立方剩餘

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Solution Strategy 思路解析

Reduce the equation modulo 9. Cubes occupy only three residue classes modulo 9, so their pairwise sums cannot produce every residue.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Modular Arithmetic 模算術** **Cubic Residues 立方剩餘**

Solution 解答

For every integer n,

$$n^3 \equiv 0, 1, -1 \pmod 9.$$

Therefore

$$x^3 + y^3 \equiv -2, -1, 0, 1, 2 \pmod 9.$$

However,

$$2020 \equiv 4 \pmod 9,$$

which is not among these possible residues. Hence the equation has no integer solution.

Final Answer 最終答案. There are no integer solutions.

Question 4 題目 4 Rational inequality | 有理不等式

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Solution Strategy 思路解析

First use $a + b \leq 2$ to obtain $ab \leq 1$. Then subtract the left-hand side from the right-hand side and factor the numerator.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 **AM–GM 算術幾何平均** **Factorisation 因式分解**

Solution 解答

By AM–GM,

$$ab \leq \left(\frac{a+b}{2}\right)^2 \leq 1.$$

Moreover,

$$\frac{2}{1+ab} - \frac{1}{1+a^2} - \frac{1}{1+b^2} = \frac{(a-b)^2(1-ab)}{(1+ab)(1+a^2)(1+b^2)} \geq 0.$$

This proves the required inequality.

Final Answer 最終答案. The inequality holds; equality occurs when $a = b$ or $ab = 1$.

Question 5 題目 5 Magic-sum tables | 定和方格

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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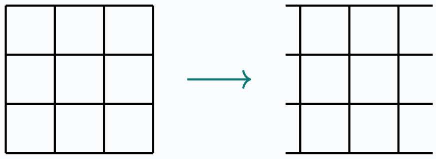
Solution Strategy 思路解析

Subtract 1 from every entry. The problem becomes counting nonnegative 3×3 tables whose row and column sums are all 2. Classify these tables according to whether an entry equal to 2 occurs.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Contingency Tables 行列和表格** **Case Classification 分類討論**



Subtract 1 from each cell Row/column sums become 2

Figure 7. The shift by 1 converts positive entries with line sum 5 into nonnegative entries with line sum 2.

Solution 解答

Subtract 1 from every cell. We count nonnegative matrices whose row sums and column sums are all 2.
If all entries are 0 or 1, each row and column has exactly two 1's. The zero positions form a permutation matrix, giving

$$3! = 6$$

tables.
Suppose an entry is 2. Its remaining row and column entries must be 0.
If there is exactly one entry equal to 2, the remaining 2×2 block must consist entirely of 1's. The position of the 2 can be chosen in

$$3 \cdot 3 = 9$$

ways.
If there is more than one entry equal to 2, the only possibility is to place three 2's, one in each row and column. Their positions form a permutation matrix, giving another

$$3! = 6$$

ways.
Therefore

$$N = 6 + 9 + 6 = 21.$$

Final Answer 最終答案. $N = 21$.

Question 6 題目 6 Angle chasing and the sine rule 丨角度追蹤與正弦定理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Determine all immediately available angles. Then compare AD/CD in two triangles using the sine rule; this fixes the two remaining angles in triangle ACD.

Olympiad Knowledge Points 奧數知識點

Euclidean Geometry 歐氏幾何 **Angle Chasing 角度追蹤** **Sine Rule 正弦定理**

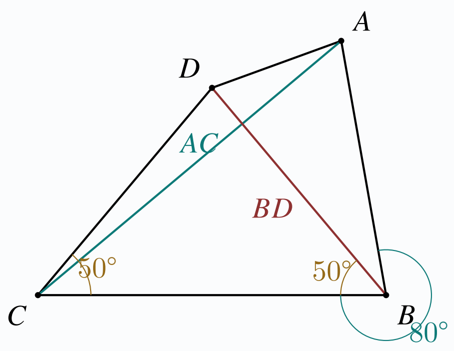

Figure 8. The two highlighted diagonals are proved perpendicular.

Solution 解答

Triangle BCD has base angles $50°$, hence

$$\angle BDC = 80°.$$

Therefore the three equal angles in the hypothesis are all $80°$.
At B,

$$\angle ABD = \angle ABC - \angle DBC = 80° - 50° = 30°.$$

Thus, in triangle ABD,

$$\angle ADB = 70°.$$

Since A and C lie on opposite sides of BD,

$$\angle ADC = 70° + 80° = 150°.$$

Let $x = \angle DAC$. Then $\angle ACD = 30° - x$. By the sine rule in triangle ABD,

$$\frac{AD}{BD} = \frac{\sin 30°}{\sin 80°}.$$

Since triangle BCD is isosceles, $BD = CD$, so

$$\frac{AD}{CD} = \frac{\sin 30°}{\sin 80°} = \frac{\sin 10°}{\sin 20°}.$$

On the other hand, the sine rule in triangle ACD gives

$$\frac{AD}{CD} = \frac{\sin(30° - x)}{\sin x}.$$

The function

$$\frac{\sin(30° - x)}{\sin x} = \frac{1}{2}\cot x - \frac{\sqrt{3}}{2}$$

is strictly decreasing for $0° < x < 30°$. Since it takes the required value at $x = 20°$, we obtain

$$x = 20°, \qquad \angle ACD = 10°.$$

Hence

$$\angle ACB = 50° - 10° = 40°.$$

Because BD makes an angle $50°$ with BC, while AC makes an angle $40°$ with CB, the angle between AC and BD is $90°$.

Final Answer 最終答案. $AC \perp BD$.

Question 7 題目 7 Repeated blocks and divisibility | 循環數字與整除

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Factor the repeated-block decimal representation. A number formed by repeating a three-digit block twice is the block multiplied by 1001.

Olympiad Knowledge Points 奧數知識點

Divisibility 整除性 | Decimal Blocks 數字分塊 | Factorisation 因式分解

Solution 解答

Let $M = \overline{XYZ}$. Then

$$N = 1000M + M = 1001M.$$

Since

$$1001 = 7 \cdot 11 \cdot 13,$$

N is always divisible by 7, 11, and 13. It is not necessarily divisible by 17.

Final Answer 最終答案. Divisibility by 17 need not hold, so the correct choice is **D**.

Question 8 題目 8 Exponential prime equation | 指數型質數方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Use parity first. Once $p = 2$ is forced, reduce $2^q + 1$ modulo 3 to restrict the prime exponent q.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Parity 奇偶性 | Modular Arithmetic 模算術

Solution 解答

If p were odd, then p^q would be odd and $r = p^q + 1$ would be an even prime. This would force $r = 2$, impossible since $p^q \geq 3$. Hence
$$p = 2.$$
Now $r = 2^q + 1$. If q is an odd prime, then
$$2^q + 1 \equiv (-1)^q + 1 \equiv 0 \pmod 3,$$
so r would be divisible by 3. The only remaining prime exponent is
$$q = 2,$$
and then $r = 5$. Therefore
$$p + q + r = 2 + 2 + 5 = 9.$$

Final Answer 最終答案. 9, which corresponds to **E** (none of the listed numerical choices).

Question 9 題目 9 Symmetric power sums | 對稱冪和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Introduce the elementary symmetric quantities $s = x + y$ and $p = xy$. Express both higher power sums in terms of s and p.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Symmetric Polynomials 對稱多項式 | Vieta's Formula 韋達定理

Solution 解答

Put
$$s = x + y, \qquad p = xy.$$
The equation $x^2 + y^2 = s$ gives
$$s^2 - 2p = s, \qquad p = \frac{s^2 - s}{2}.$$
Also,
$$x^3 + y^3 = s^3 - 3ps = s.$$
Substituting the expression for p gives
$$s^3 - \frac{3}{2}s(s^2 - s) = s,$$

so

$$s(s-1)(s-2)=0.$$

If $s=0$, then $p=0$, giving $(x,y)=(0,0)$. If $s=1$, then $p=0$, giving $(0,1)$ and $(1,0)$. If $s=2$, then $p=1$, giving $(1,1)$. Thus there are four ordered pairs.

Final Answer 最終答案. $N=4$, so the correct choice is **B**.

Question 10 題目 10 Symmetric identities | 對稱恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Use the first condition to determine $ab+bc+ca$. Squaring this quantity gives $a^2b^2+b^2c^2+c^2a^2$, which can then be inserted into the square of $a^2+b^2+c^2$.

Olympiad Knowledge Points 奧數知識點

Algebraic Identities 代數恆等式 **Symmetric Sums 對稱和**

Solution 解答

From

$$(a+b+c)^2=0$$

we obtain

$$ab+bc+ca=-2.$$

Moreover,

$$(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c),$$

so

$$a^2b^2+b^2c^2+c^2a^2=4.$$

Finally,

$$16=(a^2+b^2+c^2)^2=a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2),$$

which yields

$$a^4+b^4+c^4=16-8=8.$$

Final Answer 最終答案. 8, so the correct choice is **B**.

Question 11 題目 11 Barycentric ratios | 重心坐標比值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Represent the point of concurrency by barycentric coordinates. Each of the three segment ratios is then one normalised barycentric coordinate.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Barycentric Coordinates 重心坐標** **Cevians 共點線**

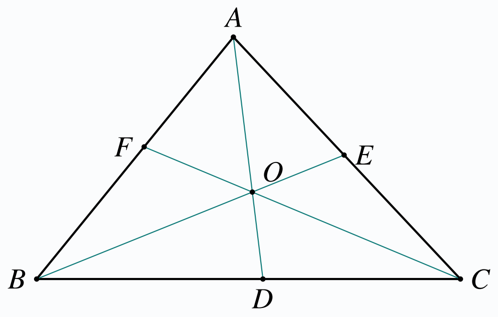

Figure 9. Three concurrent cevians in triangle ABC.

Solution 解答

Let the barycentric coordinates of O be

$$O = (\alpha : \beta : \gamma), \qquad \alpha, \beta, \gamma > 0.$$

Along the cevian AD,

$$O = \frac{\alpha}{\alpha+\beta+\gamma}A + \frac{\beta+\gamma}{\alpha+\beta+\gamma}D,$$

so

$$\frac{DO}{DA} = \frac{\alpha}{\alpha+\beta+\gamma}.$$

Similarly,

$$\frac{EO}{EB} = \frac{\beta}{\alpha+\beta+\gamma}, \qquad \frac{FO}{FC} = \frac{\gamma}{\alpha+\beta+\gamma}.$$

Adding the three ratios gives 1.

Final Answer 最終答案. 1, so the correct choice is **D.**

Question 12 題目 12 Prime powers and squares 丨質數冪與平方數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Write the perfect square as n^2 and factor $n^2 - 4$. The two factors differ by 4, which strongly restricts a product that is a power of one prime.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Prime Powers 質數冪 | Difference of Squares 平方差

Editorial Note 編按

As printed, the problem has two solutions: $(p, m) = (5, 1)$ and $(2, 5)$. Hence $p + m$ is not uniquely determined. If the intended additional condition is $m > 1$ (or that m is prime), the expected answer is 7.

Solution 解答

Let

$$n^2 = p^m + 4.$$

Then

$$(n-2)(n+2) = p^m.$$

If p is odd, both factors are odd and their greatest common divisor divides 4, hence it is 1. Therefore the smaller factor is 1:

$$n - 2 = 1, \qquad n = 3, \qquad p^m = 5.$$

Thus $(p, m) = (5, 1)$.
If $p = 2$, both $n - 2$ and $n + 2$ are powers of 2. Since they differ by 4, the only possibility is

$$n - 2 = 4, \qquad n + 2 = 8.$$

Hence $n = 6$ and

$$2^m = 32, \qquad m = 5.$$

Thus $(p, m) = (2, 5)$.

Final Answer 最終答案. As printed, $p + m \in \{6, 7\}$. Under the likely intended condition $m > 1$, $p + m = 7$.

Question 13 題目 13 Symmetric rational equations 丨對稱分式方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Introduce $s = a+b$ and $p = ab$. Both rational expressions then become simple symmetric functions of s and p.

Olympiad Knowledge Points 奧數知識點

Symmetric Algebra 對稱代數 **Vieta's Formula 韋達定理**

Solution 解答

Since the denominators are nonzero, put

$$s = a + b, \qquad p = ab.$$

The first equation gives

$$\frac{a^2 + b^2}{ab} = 3,$$

so

$$s^2 - 2p = 3p, \qquad s^2 = 5p.$$

The second equation gives

$$\frac{a^3 + b^3}{ab} = 10.$$

Now

$$a^3 + b^3 = s^3 - 3ps = s(s^2 - 3p) = 2sp,$$

therefore $2s = 10$, so $s = 5$ and $p = 5$. Thus a, b are the roots of

$$t^2 - 5t + 5 = 0.$$

Hence

$$\{a, b\} = \left\{ \frac{5 + \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2} \right\}.$$

Final Answer 最終答案. $(a, b) = \left(\frac{5 + \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2} \right)$ or the reversed pair.

Question 14 題目 14 Sums of three primes 丨三質數和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Check the interpretation carefully. If repeated primes are allowed, the requested number does not exist among the two-digit integers; if the primes are intended to be distinct, the first obstruction occurs at 11.

Olympiad Knowledge Points 奥數知識點

Prime Representations 質數表示 **Problem Interpretation 題意辨析**

Editorial Note 編按

The printed question does not say that the three primes must be distinct. With the usual convention that repetitions are allowed, every two-digit integer is a sum of three primes. The likely intended wording is "three distinct primes", for which the answer is 11.

Solution 解答

If repetitions are allowed, every two-digit integer is representable. For an even integer n, use $n = 2+(n-2)$; for an odd integer n, use $n = 3+(n-3)$. It remains only to verify that every even integer from 8 to 96 is a sum of two primes. The following compact list supplies such representations:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
8 &=& 3+5 & 38 &=& 7+31 & 68 &=& 7+61\\
10 &=& 3+7 & 40 &=& 3+37 & 70 &=& 3+67\\
12 &=& 5+7 & 42 &=& 5+37 & 72 &=& 5+67\\
14 &=& 3+11 & 44 &=& 3+41 & 74 &=& 3+71\\
16 &=& 3+13 & 46 &=& 3+43 & 76 &=& 3+73\\
18 &=& 5+13 & 48 &=& 5+43 & 78 &=& 5+73\\
20 &=& 3+17 & 50 &=& 3+47 & 80 &=& 7+73\\
22 &=& 3+19 & 52 &=& 5+47 & 82 &=& 3+79\\
24 &=& 5+19 & 54 &=& 7+47 & 84 &=& 5+79\\
26 &=& 3+23 & 56 &=& 3+53 & 86 &=& 3+83\\
28 &=& 5+23 & 58 &=& 5+53 & 88 &=& 5+83\\
30 &=& 7+23 & 60 &=& 7+53 & 90 &=& 7+83\\
32 &=& 3+29 & 62 &=& 3+59 & 92 &=& 3+89\\
34 &=& 3+31 & 64 &=& 3+61 & 94 &=& 5+89\\
36 &=& 5+31 & 66 &=& 5+61 & 96 &=& 7+89
\end{array}$$

Hence no two-digit counterexample exists.
If the primes must be distinct, then

$$10 = 2+3+5,$$

whereas 11 cannot be written as a sum of three distinct primes. Thus the intended answer would be 11.

Final Answer 最終答案. As printed, no such two-digit number exists. If "distinct primes" was intended, the answer is 11.

Question 15 題目 15 Cubic factorisation | 三次式因式分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Treat the expression as a homogeneous polynomial and look for the obvious vanishing factors: it is zero when $a = 0$, $b = 0$, $x = 0$, $y = 0$, $a = b$, or $x = y$.

Olympiad Knowledge Points 奧數知識點

Factor Theorem 因式定理 **Homogeneous Polynomials 齊次多項式**

Solution 解答

Expand the two cubes, but keep the pure-cube terms and the mixed terms grouped:

$$\begin{aligned}(ay + bx)^3 - (ax + by)^3 = a^3(y^3 - x^3) + b^3(x^3 - y^3)\\ + 3a^2b\,xy(y - x) + 3ab^2\,xy(x - y).\end{aligned}$$

After adding $(a^3 - b^3)(x^3 - y^3)$, the pure-cube terms cancel completely. Hence the whole expression equals

$$\begin{aligned}&3a^2b\,xy(y - x) + 3ab^2\,xy(x - y)\\ &= 3abxy\big[a(y - x) + b(x - y)\big]\\ &= 3abxy(a - b)(y - x)\\ &= -3abxy(a - b)(x - y).\end{aligned}$$

This also makes all six expected linear factors visible.

Final Answer 最終答案. $\boxed{-3abxy(a - b)(x - y)}$.

Question 16 題目 16 Real algebraic system | 實數代數方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Let $u = ab$ and $t = b^2$. The two equations become a simple sum-and-product system in u and t.

Olympiad Knowledge Points 奧數知識點

Algebraic Substitution 代數代換 | Quadratic Equation 二次方程

Solution 解答

Put

$$u = ab, \qquad t = b^2 \geq 0.$$

Then the equations become

$$ut = -135, \qquad u + t = -6.$$

Thus

$$t(-6 - t) = -135,$$

so

$$t^2 + 6t - 135 = 0, \qquad (t - 9)(t + 15) = 0.$$

Since $t \geq 0$, we have $t = 9$ and $u = -15$. Therefore $b = \pm 3$ and $a = u/b$, giving

$$(a, b) = (-5, 3), \qquad (5, -3).$$

Final Answer 最終答案. $(a, b) = (-5, 3)$ or $(5, -3)$.

Question 17 題目 17 Sum of two squares | 兩平方和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Search near $\sqrt{2020} \approx 45$ and use the factorisation $2020 = 4 \cdot 505$ to look for even solutions.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Sum of Two Squares 兩平方和

Solution 解答

A direct calculation gives

$$16^2 + 42^2 = 256 + 1764 = 2020.$$

Thus $(16, 42)$ is a valid pair. Another valid pair is $(24, 38)$.

Final Answer 最終答案. For example, $(x, y) = (16, 42)$.

Question 18 題目 18 Affine geometry and line ratios | 仿射幾何與截線比

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

The problem is affine, so choose convenient coordinates. Parameterise the line BF and find the parameter values of its intersections with AX and AY.

Olympiad Knowledge Points 奧數知識點

Affine Geometry 仿射幾何 | **Coordinate Geometry 坐標幾何** | **Section Ratios 分點比**

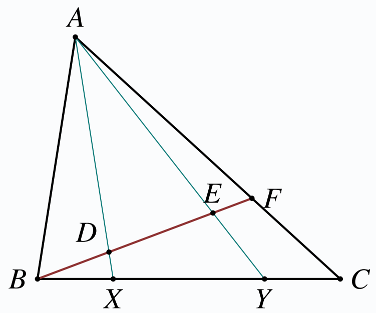

Figure 10. The transversal BF meets the two cevians AX and AY at D and E.

Solution 解答

Choose coordinates

$$B = (0,0), \qquad C = (4,0), \qquad X = (1,0), \qquad Y = (3,0), \qquad A = (0,h).$$

Since $AF : FC = 2 : 1$,

$$F = \frac{1}{3}A + \frac{2}{3}C = \left(\frac{8}{3}, \frac{h}{3}\right).$$

A point on BF has the form

$$P(t) = tF = \left(\frac{8t}{3}, \frac{ht}{3}\right), \qquad 0 \le t \le 1.$$

On AX, write a point as $(s, h(1-s))$. Comparing the y-coordinates with $P(t)$ gives $s = 1 - t/3$, while the x-coordinates give

$$1 - \frac{t}{3} = \frac{8t}{3}, \qquad t_D = \frac{1}{3}.$$

Similarly, a point on AY has coordinates $(3s, h(1-s))$. Thus

$$3 - t = \frac{8t}{3}, \qquad t_E = \frac{9}{11}.$$

Therefore, as fractions of BF,

$$BD = \frac{1}{3}BF, \quad DE = \left(\frac{9}{11} - \frac{1}{3}\right)BF = \frac{16}{33}BF, \quad EF = \frac{2}{11}BF = \frac{6}{33}BF.$$

Hence

$$BD : DE : EF = 11 : 16 : 6.$$

Final Answer 最終答案. $BD : DE : EF = 11 : 16 : 6.$

Question 19 題目 19 Integer roots and Vieta | 整數根與韋達定理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Call the roots m, n and use Vieta's formula. Eliminating k produces a factorisation in m, n, reducing the problem to a short divisibility check.

Olympiad Knowledge Points 奧數知識點

Diophantine Equations 丟番圖方程 | Vieta's Formula 韋達定理 | Factorisation 因式分解

Solution 解答

Let the positive integer roots be $m \le n$. By Vieta's formula,

$$m + n = \frac{6(3k-1)}{k^2-1}, \qquad mn = \frac{72}{k^2-1}.$$

Hence

$$3k - 1 = 12\left(\frac{1}{m} + \frac{1}{n}\right),$$

so

$$k = \frac{mn + 12m + 12n}{3mn}.$$

Substituting this into $k^2 - 1 = 72/(mn)$ and simplifying gives

$$(mn - 6m + 3n)(mn + 3m - 6n) = 0.$$

If

$$mn + 3m - 6n = 0,$$

then

$$n = \frac{3m}{6-m}.$$

Since $m \le n$ and $n > 0$, it is enough to test $m = 1, 2, 3, 4, 5$. The integer solutions are

$$(m, n) = (3, 3), (4, 6), (5, 15).$$

The other factor gives the same pairs after interchanging m, n.
The corresponding values of k are

$$k = 3, \qquad k = 2, \qquad k = \frac{7}{5}.$$

Therefore

$$S = 3 + 2 + \frac{7}{5} = \frac{32}{5}.$$

Final Answer 最終答案. $S = \dfrac{32}{5}$.

2020 | Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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本公開讀本不重新封裝原卷。請前往教青局比賽題庫，按學年及組別開啓原卷，並以該原卷為準。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Derangements | 錯排

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

An operation is a permutation of the five coins in which no coin remains with its original owner. Thus the problem asks for the number of derangements of five objects.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 | Derangements 錯排 | Inclusion–Exclusion 容斥原理

Solution 解答

By inclusion–exclusion, the number of permutations of five coins with no fixed point is

$$\begin{aligned} !5 &= 5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5} \\ &= 120 - 120 + 60 - 20 + 5 - 1 \\ &= 44. \end{aligned}$$

Final Answer 最終答案. 44 operations.

Question 2 題目 2 Nonlinear recurrence and induction | 非線性遞推與歸納

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Prove a slightly stronger statement with an additional constant 2. The recurrence becomes especially convenient after writing $x_k = t + 2$.

Olympiad Knowledge Points 奧數知識點

Mathematical Induction 數學歸納法 | Recurrence Relations 遞推數列 | Strengthened Induction 强化命題

Solution 解答

We prove the stronger assertion

$$x_{k+1} \geq 3^{2^k} + 2 \qquad (k \geq 0).$$

For $k = 0$,

$$x_1 = 5 = 3 + 2.$$

Assume

$$x_k \geq 3^{2^{k-1}} + 2.$$

Put $t = 3^{2^{k-1}}$. Since the function $x^2 - 3x + 3$ is increasing for $x \geq 2$, we have

$$\begin{aligned} x_{k+1} &\geq (t+2)^2 - 3(t+2) + 3 \\ &= t^2 + t + 1 \\ &\geq t^2 + 2 \\ &= 3^{2^k} + 2. \end{aligned}$$

This completes the induction, and in particular

$$x_{k+1} \geq 3^{2^k}.$$

Final Answer 最終答案. The inequality holds for every $k \geq 0$.

Question 3 題目 3 Cauchy inequality | 柯西不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Pair adjacent terms on the left and apply the Cauchy–Schwarz inequality. The condition $a + b + c = 2$ turns each paired numerator into the remaining variable. A simple telescoping identity then symmetrises the result.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Cauchy–Schwarz 柯西不等式 | Cyclic Symmetrisation 循環對稱化

Solution 解答

By Cauchy–Schwarz,

$$\frac{(a-1)^2}{b} + \frac{(b-1)^2}{c} \geq \frac{(2-a-b)^2}{b+c} = \frac{c^2}{b+c}.$$

Cyclically,

$$\frac{(b-1)^2}{c} + \frac{(c-1)^2}{a} \ge \frac{a^2}{c+a},$$
$$\frac{(c-1)^2}{a} + \frac{(a-1)^2}{b} \ge \frac{b^2}{a+b}.$$

Adding these inequalities gives

$$2L \ge \frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a},$$

where L denotes the left-hand side of the desired inequality.
Now

$$\left(\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a}\right) - \left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right)$$
$$= (b-a) + (c-b) + (a-c) = 0.$$

Thus each parenthesis equals one half of

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}.$$

Consequently

$$2L \ge \frac{1}{2}\left(\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}\right),$$

which is exactly the required result.

Final Answer 最終答案. The inequality is proved.

Question 4 題目 4 Cyclic geometry and incircles | 圓內接四邊形與內切圓

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let the common angle be α and resolve the other angles relative to the chord BD. Compute the height of the upper horizontal tangent to each incircle using the inradius formula $r = 4R\prod\sin(A/2)$; the two heights turn out to be equal.

Olympiad Knowledge Points 奧數知識點

- **Cyclic Geometry 圓內接幾何**
- **Incentres 內心**
- **Half-angle Formula 半角公式**
- **Support Lines 支撐線**

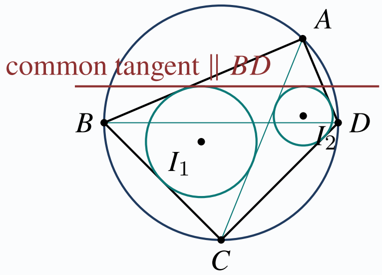

Figure 11. A representative configuration. The red line is tangent to both incircles and parallel to BD.

Solution 解答

Let
$$\alpha = \angle BAC = \angle CAD, \quad \beta = \angle ABD, \quad \gamma = \angle BDA.$$
Since $ABCD$ is cyclic,
$$\angle CBD = \angle CAD = \alpha, \qquad \angle BDC = \angle BAC = \alpha.$$
Also
$$2\alpha + \beta + \gamma = 180^\circ.$$
Thus the angles of triangle ABC are
$$\alpha, \quad \alpha + \beta, \quad \gamma,$$
and those of triangle ACD are
$$\alpha, \quad \beta, \quad \alpha + \gamma.$$
Let R be the common circumradius and let r_1, r_2 be the two inradii. Using
$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2},$$
we obtain
$$r_1 = 4R \sin\frac{\alpha}{2} \sin\frac{\alpha+\beta}{2} \sin\frac{\gamma}{2},$$
$$r_2 = 4R \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\alpha+\gamma}{2}.$$
Take BD as a horizontal reference line. At B, the two rays BA and BC make angles β above and α below BD. Therefore the signed height h_1 of I_1 above BD is
$$h_1 = r_1 \frac{\sin\frac{\beta-\alpha}{2}}{\sin\frac{\alpha+\beta}{2}}.$$
The height of the upper horizontal tangent to the first incircle is
$$\begin{aligned} h_1 + r_1 &= r_1 \frac{\sin\frac{\beta-\alpha}{2} + \sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}} \\ &= \frac{2r_1 \sin(\beta/2)\cos(\alpha/2)}{\sin((\alpha+\beta)/2)} \\ &= 4R \sin\alpha \sin\frac{\beta}{2} \sin\frac{\gamma}{2}. \end{aligned}$$

Repeating the same calculation at D gives the height of the upper horizontal tangent to the second incircle:

$$\begin{aligned} h_2 + r_2 &= \frac{2r_2 \sin(\gamma/2)\cos(\alpha/2)}{\sin((\alpha+\gamma)/2)} \\ &= 4R\sin\alpha\sin\frac{\beta}{2}\sin\frac{\gamma}{2}. \end{aligned}$$

The two heights are equal. Hence the same line parallel to BD is tangent externally to both incircles.

Final Answer 最終答案. An external common tangent of the two incircles is parallel to BD.

Question 5 題目 5 Cubic identities and integer equations | 立方恆等式與整數方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

For part (a), write the four integers as $n, n+1, n+2, n+3$ and factor the resulting polynomial. For part (b), introduce a third term -1 and apply the identity for $u^3+v^3+w^3-3uvw$.

Olympiad Knowledge Points 奧數知識點

Integer Equations 整數方程 **Cubic Identity 立方恆等式** **Factorisation 因式分解**

Solution 解答

For part (a), the required equation is

$$(n+3)^3 = n^3 + (n+1)^3 + (n+2)^3.$$

Rearranging and factorising,

$$(n+3)^3 - n^3 - (n+1)^3 - (n+2)^3 = -2(n-3)(n^2+3n+3).$$

Since the quadratic factor is positive, $n=3$. Thus the integers are

$$3, 4, 5, 6,$$

and indeed

$$6^3 = 3^3 + 4^3 + 5^3.$$

For part (b), write the equation as

$$x^3 + y^3 + (-1)^3 - 3xy(-1) = 0.$$

Using

$$u^3+v^3+w^3-3uvw = (u+v+w)(u^2+v^2+w^2-uv-vw-wu),$$

we get

$$(x+y-1)(x^2+y^2+1-xy+x+y)=0.$$

The second factor can vanish only at $x=y=-1$. Otherwise

$$x+y=1.$$

Therefore all integer solutions are

$$(x,y)=(t,1-t)\quad (t\in\mathbb{Z}),$$

together with

$$(x,y)=(-1,-1).$$

Final Answer 最終答案. (a) $3,4,5,6.$ (b) $(t,1-t)$ for $t\in\mathbb{Z}$, and $(-1,-1)$.

Question 6 題目 6 Alternating square sum | 平方差交錯和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Pair the alternating terms as $(2j)^2-(2j-1)^2$. Each pair collapses to the linear expression $4j-1$, so the original long alternating sum becomes an elementary arithmetic-series sum.

Olympiad Knowledge Points 奧數知識點

Arithmetic Series 等差級數 **Difference of Squares 平方差**

Solution 解答

For $j=1,2,\ldots,1010$,

$$(2j)^2-(2j-1)^2=4j-1.$$

Therefore

$$\begin{aligned}N&=\sum_{j=1}^{1010}(4j-1)\\&=4\cdot\frac{1010\cdot 1011}{2}-1010\\&=2,041,210.\end{aligned}$$

This is outside all four numerical ranges listed.

Final Answer 最終答案. $N=2,041,210$, so the correct choice is **E.**

Question 7 題目 7 Partial fractions | 部分分式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use

$$x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$$

and compare coefficients after clearing denominators.

Olympiad Knowledge Points 奧數知識點

Partial Fractions 部分分式 **Coefficient Comparison 比較係數**

Solution 解答

Clearing denominators gives

$$6x^3 + 10x = (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1).$$

Comparing coefficients yields

$$A = 3, \qquad B = -2, \qquad C = 3, \qquad D = 2.$$

Hence

$$A + B + 3C + 3D = 3 - 2 + 9 + 6 = 16.$$

Final Answer 最終答案. 16, so the correct choice is **D**.

Question 8 題目 8 Cyclic rational identity | 循環分式恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use $c = 1/(ab)$ to rewrite all three denominators as scalar multiples of one common expression.

Olympiad Knowledge Points 奧數知識點

Algebraic Identity 代數恆等式 **Cyclic Expressions 循環式**

Solution 解答

Let

$$D = 1 + a + ab.$$

Since $abc = 1$,

$$bc + b + 1 = \frac{1}{a} + b + 1 = \frac{D}{a},$$

and

$$ac + c + 1 = \frac{1}{b} + \frac{1}{ab} + 1 = \frac{D}{ab}.$$

Therefore the required sum is

$$\frac{1}{D} + \frac{a}{D} + \frac{ab}{D} = 1.$$

Final Answer 最終答案. 1, so the correct choice is **C**.

Question 9 題目 9 Four-cube identity | 四立方恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Expand only far enough to observe cancellation. All pure cubes and all terms containing a squared variable cancel; only the fully mixed term remains.

Olympiad Knowledge Points 奧數知識點

Algebraic Identities 代數恆等式 | Symmetric Cancellation 對稱消去

Solution 解答

Expanding and collecting like terms, all terms except the product abc cancel. The remaining coefficient is 24, so

$$(a+b+c)^3 + (b-a-c)^3 + (c-a-b)^3 + (a-b-c)^3 = 24abc.$$

Final Answer 最終答案. $24abc$, so the correct choice is **D**.

Question 10 題目 10 Polynomial optimisation | 多項式最值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Differentiate, or factor the difference between $f(x)$ and the candidate minimum.

Olympiad Knowledge Points 奧數知識點

Optimisation 最值 | **Calculus 微積分** | **Polynomial Factorisation 多項式分解**

Solution 解答

We have

$$f'(x) = 4x^3 - 4 = 4(x-1)(x^2+x+1).$$

The only real critical point is $x = 1$, and

$$f(1) = 1 - 4 = -3.$$

Since $f(x) \to +\infty$ as $|x| \to \infty$, this is the global minimum.

Final Answer 最終答案. -3.

Question 11 題目 11 Terminal digits | 末位數字

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Extract the power of 10. The requested digits are then the tens and hundreds digits of $2 \cdot 9^{2018}$, which can be found modulo 1000.

Olympiad Knowledge Points 奧數知識點

Modular Arithmetic 模算術 | **Last Digits 末位數字** | **Euler's Theorem 歐拉定理**

Solution 解答

We have

$$2^{2020}5^{2019}9^{2018} = 2 \cdot 9^{2018} \cdot 10^{2019}.$$

Thus $a_{2019}, a_{2020}, a_{2021}$ are the units, tens, and hundreds digits of $2 \cdot 9^{2018}$.
Since $\varphi(1000) = 400$,

$$9^{2018} \equiv 9^{18} \pmod{1000}.$$

Repeated squaring gives

$$9^{18} \equiv 121 \pmod{1000},$$

so

$$2 \cdot 9^{2018} \equiv 242 \pmod{1000}.$$

Therefore

$$a_{2020} = 4, \qquad a_{2021} = 2,$$

and their sum is 6.

Final Answer 最終答案. $a_{2020} + a_{2021} = 6$.

Question 12 題目 12 Cubic factorisation | 三次因式分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Both parts are manifestations of the same three-variable cubic identity.

Olympiad Knowledge Points 奧數知識點

Cubic Identity 立方恆等式 **Factorisation 因式分解**

Solution 解答

For part (a),

$$\begin{aligned} f(-a-b) &= -(a+b)^3 + 3ab(a+b) + a^3 + b^3 \\ &= 0. \end{aligned}$$

For part (b),

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Final Answer 最終答案. (a) 0. (b) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Question 13 題目 13 Five cubes representation 丨五個立方數表示

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use two large equal cubes and correct the remaining small difference with three small signed cubes.

Olympiad Knowledge Points 奧數知識點

Constructive Number Theory 構造數論 | Signed Cubes 帶符號立方數

Solution 解答

One convenient representation is

$$2020 = 10^3 + 10^3 + 3^3 + 1^3 + (-2)^3.$$

Indeed,

$$1000 + 1000 + 27 + 1 - 8 = 2020.$$

Final Answer 最終答案. $2020 = 10^3 + 10^3 + 3^3 + 1^3 + (-2)^3$.

Question 14 題目 14 Real algebraic system 丨實數代數方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Set $u = ab$ and $t = b^2$ so that both original equations become relations only in u and t. Their product and sum give a quadratic for t; after determining $t = b^2$, recover $u = ab$ and hence the two possible signs of b.

Olympiad Knowledge Points 奧數知識點

Algebraic Substitution 代數代換 | Quadratic Equations 二次方程

Solution 解答

Let

$$u = ab, \qquad t = b^2.$$

Then

$$ut = -135, \qquad u + t = -6.$$

Thus

$$t^2 + 6t - 135 = 0,$$

so $t = 9$. Hence $u = -15$ and

$$(a, b) = (-5, 3), \qquad (5, -3).$$

Final Answer 最終答案. $(a, b) = (-5, 3)$ or $(5, -3)$.

Question 15 題目 15 Minimax approximation | 極小極大逼近

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use a discrete second difference at $x = 0, 1, 2$ to obtain a universal lower bound. Then construct a quadratic whose extreme values alternate between the two bounds.

Olympiad Knowledge Points 奧數知識點

Minimax Principle 極小極大原理 | **Finite Differences 有限差分** | **Equioscillation 等振盪**

Solution 解答

Let

$$q(x) = 2x^2 + bx + c, \qquad M = \max_{0 \le x \le 2} |q(x)|.$$

A direct calculation gives

$$q(0) - 2q(1) + q(2) = 4.$$

Therefore

$$4 \le |q(0)| + 2|q(1)| + |q(2)| \le 4M,$$

so

$$M \ge 1.$$

Now choose

$$b = -4, \qquad c = 1.$$

Then

$$q(x) = 2(x - 1)^2 - 1.$$

For $0 \le x \le 2$, this expression lies between -1 and 1. Hence $M(-4, 1) = 1$, attaining the lower bound.

Final Answer 最終答案. $\min M(b, c) = 1$.

Question 16 題目 16 Interchanging infinite sums 丨交換無窮和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

All terms are positive, so interchange the order of summation. Each inner sum is a geometric series and the resulting finite sum telescopes.

Olympiad Knowledge Points 奧數知識點

Infinite Series 無窮級數 | Geometric Series 等比級數 | Telescoping Sum 望遠鏡求和

Solution 解答

Since all terms are positive,

$$
\begin{aligned}
S &= \sum_{n=2}^{\infty}\sum_{k=2}^{2020}\frac{1}{k^n} \\
&= \sum_{k=2}^{2020}\sum_{n=2}^{\infty}\frac{1}{k^n} \\
&= \sum_{k=2}^{2020}\frac{1/k^2}{1-1/k} \\
&= \sum_{k=2}^{2020}\frac{1}{k(k-1)} \\
&= \sum_{k=2}^{2020}\left(\frac{1}{k-1}-\frac{1}{k}\right) \\
&= 1-\frac{1}{2020} \\
&= \frac{2019}{2020}.
\end{aligned}
$$

Final Answer 最終答案. $S = \dfrac{2019}{2020}$.

2021 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Integer systems 丨整數方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Shift each variable by 1. The three nonlinear equations then become a very symmetric multiplicative system, which can be split according to whether the shifted variables are zero.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Symmetric Systems 對稱方程組 | Variable Shift 變量平移

Solution 解答

Set

$$a = x + 1, \qquad b = y + 1, \qquad c = z + 1.$$

The given equations are equivalent to

$$ab = c, \qquad bc = a, \qquad ca = b.$$

If one of a, b, c is zero, then the equations force

$$a = b = c = 0,$$

which gives $(x, y, z) = (-1, -1, -1)$.
Now suppose that $abc \neq 0$. From $ab = c$ and $bc = a$,

$$ab^2 = a,$$

so $b^2 = 1$. Similarly, $a^2 = c^2 = 1$. We may choose $a, b \in \{1, -1\}$ freely, after which $c = ab$. This gives four further triples.
Hence the total number of integer triples is

$$1 + 4 = 5.$$

Final Answer 最終答案. 5, so the correct choice is **E**.

Question 2 題目 2 Counting triangles 丨三角形計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Classify the triangles by orientation and side length. Count upward-pointing triangles first, then downward-pointing triangles.

Olympiad Knowledge Points 奧數知識點

Combinatorial Geometry 組合幾何 **Systematic Counting 系統計數** **Triangular Grids 三角形網格**

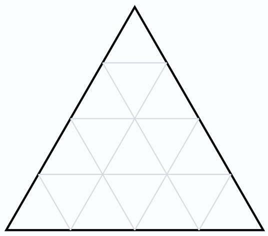

Figure 12. The triangular grid has side length 4.

Solution 解答

For upward-pointing triangles, the numbers with side lengths $1, 2, 3, 4$ are respectively

$$10, 6, 3, 1.$$

Thus there are

$$10 + 6 + 3 + 1 = 20$$

upward-pointing triangles.
For downward-pointing triangles, there are 6 of side length 1 and 1 of side length 2. Hence there are

$$6 + 1 = 7$$

downward-pointing triangles.
Therefore the total is

$$20 + 7 = 27.$$

Final Answer 最終答案. 27, so the correct choice is **C**.

Question 3 題目 3 Factor comparison 丨因式比較

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Compare the product with 111^2. Expanding isolates the sign of $a-b$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Sign Analysis 符號分析 | Expansion 展開

Solution 解答

Since $111^2 = 12321$, expansion gives

$$12345 = 12321 + 111(a-b) - ab.$$

Hence

$$111(a-b) - ab = 24 > 0.$$

If $a \le b$, then $111(a-b) \le 0$ and $-ab < 0$, so the left-hand side would be negative. This is impossible. Therefore

$$a > b.$$

Final Answer 最終答案. $a > b$, so the correct choice is **A**.

Question 4 題目 4 Aliquot sums | 真因數和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Editorial Note 編按

The English wording says “all positive divisors”, but the printed example $D(6) = 1+2+3 = 6$ excludes 6 itself. Accordingly, $D(n)$ is interpreted as the sum of the positive proper divisors, consistently with the example.

Solution Strategy 思路解析

Separate odd and even n. For odd n, use the parity of the divisor-sum function. For even n, the proper divisor $n/2$ gives a strong upper bound, leaving a short case analysis.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Divisor Sums 因數和 | Parity of $\sigma(n)$ 因數和奇偶性 | Case Analysis 分類討論

Solution 解答

Write $\sigma(n)$ for the sum of all positive divisors of n. The condition is

$$\sigma(n) = n + 42.$$

Case 1: n is odd. Then $\sigma(n)$ is odd. The divisors of a nonsquare can be paired into distinct pairs $d, n/d$, whose sums are even; hence $\sigma(n)$ is odd only when n is a square. Write $n = m^2$.
Since m is a proper divisor, $1 + m \le 42$, so $m \le 41$. If m is prime, then

$$D(m^2) = 1 + m = 42,$$

so $m = 41$ and

$$n = 41^2 = 1681.$$

If m is composite and odd, then either $m = 9$, giving

$$D(81) = 1 + 3 + 9 + 27 = 40,$$

or $m \ge 15$. In the latter case, if p is the smallest prime divisor of m, then $m/p \ge 3$, and the distinct proper divisors

$$1, \quad p, \quad m, \quad \frac{m^2}{p}$$

have sum at least

$$1 + 3 + m + 3m = 4m + 4 \ge 64 > 42.$$

Thus the only odd solution is 1681.
Case 2: n is even. Since $n/2$ is a proper divisor, $n/2 \le 42$, so $n \le 84$. Write

$$n = 2^a m,$$

where m is odd. Then

$$D(n) = (2^{a+1} - 1)\sigma(m) - 2^a m = 42.$$

The possibilities are short:

a	restriction on m	result
1	$m \in \{3, 9, 15, 21, 27, 33, 39\}$	$m = 15$ only
2	$m \in \{7, 21\}$	none
3	$m \le 10,\ m \equiv 6 \pmod{15}$	none
4	$m \le 5,\ m \equiv 9 \pmod{31}$	none
5	$m = 1$	$D(32) = 31$
6	$m = 1$	$D(64) = 63$

For $a = 1$ and $m = 15$,

$$n = 30, \qquad D(30) = 1 + 2 + 3 + 5 + 6 + 10 + 15 = 42.$$

Therefore the two solutions are

$$30 \quad \text{and} \quad 1681.$$

Final Answer 最終答案. 2, so the correct choice is **B**.

Question 5 題目 5 Areas and midpoints 丨面積與中點

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Express each corner region as half the sum of the two adjacent triangles with vertex P. Opposite corner regions then have equal total area.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Area Chasing 面積追蹤** **Midpoint Areas 中點面積**

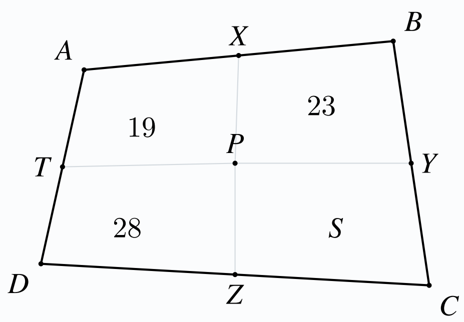

Figure 13. Each corner region is formed from halves of two adjacent triangles with vertex P.

Solution 解答

Let

$$u = [PAB], \quad v = [PBC], \quad w = [PCD], \quad t = [PDA].$$

Because X, Y, Z, T are midpoints,

$$[AXPT] = \frac{u+t}{2} = 19,$$

$$[BYPX] = \frac{u+v}{2} = 23,$$

$$[CZPY] = \frac{v+w}{2} = S,$$

$$[DTPZ] = \frac{w+t}{2} = 28.$$

Adding opposite regions,

$$19 + S = \frac{u+v+w+t}{2} = 23 + 28.$$

Therefore

$$S = 51 - 19 = 32.$$

Final Answer 最終答案. $S = 32$.

Question 6 題目 6 Telescoping product | 望遠鏡乘積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Factor each term into two consecutive ratios. The product then telescopes from both ends.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Telescoping Products 望遠鏡乘積** **Factorisation 因式分解**

Solution 解答

For $k \geq 2$,

$$1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2} = \frac{k-1}{k} \cdot \frac{k+1}{k}.$$

Hence

$$\begin{aligned} P &= \left(\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{2020}{2021}\right)\left(\frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2022}{2021}\right) \\ &= \frac{1}{2021} \cdot \frac{2022}{2} = \frac{1011}{2021}. \end{aligned}$$

Final Answer 最終答案. $P = \dfrac{1011}{2021}$.

Question 7 題目 7 Linear equations | 一次方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Substitute successively to express b and c in terms of a, then use the last equation.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Linear Systems 一次方程組** **Substitution 代入法**

Solution 解答

We have

$$c = 3(2a + 1) + 2 = 6a + 5.$$

Thus

$$120 - a = 4(6a + 5) = 24a + 20,$$

so

$$100 = 25a, \qquad a = 4.$$

Then

$$b = 9, \qquad c = 29,$$

and therefore

$$a + b + c = 4 + 9 + 29 = 42.$$

Final Answer 最終答案. 42.

Question 8 題目 8 Directed angles | 有向角

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Track the directed inclination of each segment relative to the horizontal. At each interior vertex, the change in direction is the corresponding exterior angle 180°− (interior angle).

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Directed Angles 有向角** **Exterior Turns 外角轉向**

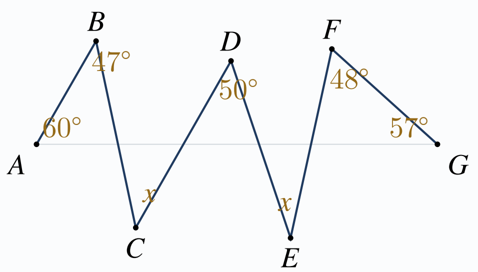

Figure 14. The alternating exterior turns determine the final inclination.

Solution 解答

Take the positive horizontal direction as 0°. Since $\angle A = 60°$, the direction of AB is

$$60°.$$

At B the path turns clockwise through $180° - 47° = 133°$, so

$$\mathrm{dir}(BC) = 60° - 133° = -73°.$$

At C it turns counterclockwise through $180° - x$, hence

$$\mathrm{dir}(CD) = -73° + 180° - x = 107° - x.$$

At D it turns clockwise through $130°$:

$$\mathrm{dir}(DE) = -23° - x.$$

At E it turns counterclockwise through $180° - x$:

$$\mathrm{dir}(EF) = 157° - 2x.$$

At F it turns clockwise through $180° - 48° = 132°$:

$$\mathrm{dir}(FG) = 25° - 2x.$$

At G, the angle between GF and the horizontal ray GA is $57°$. Since the drawing gives $2x > 25°$,

$$2x - 25° = 57°.$$

Thus

$$x = 41°.$$

Final Answer 最終答案. $x = 41°$.

Question 9 題目 9 Optimising a right triangle | 直角三角形最值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use $2S = ab$ to transform the condition into a factored constraint. A one-variable product constraint then yields the minimum by AM–GM.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Optimisation 最值** **AM–GM 算術幾何平均** **Factorisation 因式分解**

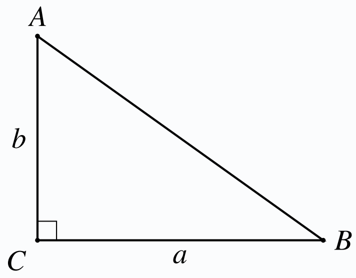

Figure 15. The area is $S = ab/2$.

Solution 解答

Since ABC is right-angled at C,

$$2S = ab.$$

The condition becomes

$$ab = 3a + 2b,$$

so

$$(a-2)(b-3) = 6.$$

Write

$$x = a-2 > 0, \qquad y = b-3 > 0.$$

Then $xy = 6$, and

$$ab = (x+2)(y+3) = 12 + 3x + 2y.$$

By AM–GM,

$$3x + 2y \geq 2\sqrt{6xy} = 12.$$

Therefore

$$ab \geq 24, \qquad S = \frac{ab}{2} \geq 12.$$

Equality occurs when $3x = 2y$ and $xy = 6$, namely $x = 2$, $y = 3$, so $a = 4$, $b = 6$.

Final Answer 最終答案. $\min S = 12$.

Question 10 題目 10 Extremal selection 丨極值選取

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Find the largest subset avoiding the required ratio. In increasing order, consecutive terms must grow by a factor strictly greater than $3/2$. A greedy construction minimises each next term and therefore maximises the possible length.

Olympiad Knowledge Points 奧數知識點

Extremal Combinatorics 極值組合 | Greedy Method 貪心法 | Pigeonhole Principle 鴿巢原理

Solution 解答

Suppose

$$a_1 < a_2 < \cdots < a_k$$

is a selected set containing no required pair. Then necessarily

$$a_{i+1} > \frac{3}{2}a_i,$$

so

$$a_{i+1} \geq \left\lfloor \frac{3a_i}{2} \right\rfloor + 1.$$

Starting with the smallest possible value $a_1 = 1$, the greedy sequence is

$$1, 2, 4, 7, 11, 17, 26, 40, 61, 92, 139, \ldots$$

The first ten terms lie in $\{1, \ldots, 130\}$, but the eleventh does not. By induction, every ratio-avoiding sequence has its ith term at least as large as the ith greedy term. Hence at most ten integers can be selected without producing the required pair.

The ten-term set

$$\{1, 2, 4, 7, 11, 17, 26, 40, 61, 92\}$$

indeed avoids the interval $1 < x/y \leq 3/2$. Therefore ten numbers are not sufficient, whereas eleven always are.

Final Answer 最終答案. $n = 11$.

Question 11 題目 11 Transforming roots | 根的變換

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use Vieta's formulas to compute the sum and product of the transformed roots directly.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Vieta's Formulas 韋達定理 | Root Transformation 根的變換

Solution 解答

By Vieta's formulas,

$$\alpha+\beta=-2, \qquad \alpha\beta=-6.$$

Also,

$$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1=-6+2+1=-3.$$

Therefore the sum of the new roots is

$$\frac{1}{\alpha-1}+\frac{1}{\beta-1}=\frac{\alpha+\beta-2}{(\alpha-1)(\beta-1)}=\frac{-4}{-3}=\frac{4}{3},$$

and their product is

$$\frac{1}{(\alpha-1)(\beta-1)}=-\frac{1}{3}.$$

Thus the equation is

$$t^2-\frac{4}{3}t-\frac{1}{3}=0,$$

or equivalently

$$3t^2-4t-1=0.$$

Final Answer 最終答案. $\boxed{3x^2-4x-1=0}$.

Question 12 題目 12 AM–GM inequality | 算術幾何平均不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Apply AM–GM directly to a,b,c to obtain a lower bound for their arithmetic mean in terms of $(abc)^{1/3}$. Because both sides are positive, cubing preserves the inequality; record the equality condition $a=b=c$ as part of the proof.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 **AM–GM 算術幾何平均** **Equality Cases 等號條件**

Solution 解答

By AM–GM,

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}.$$

Cubing both sides gives

$$(a+b+c)^3\geq 27abc.$$

Equality holds if and only if

$$a=b=c.$$

□

Final Answer 最終答案. The inequality is proved, with equality exactly when $a = b = c$.

Question 13 題目 13 Ptolemy inequality and area | 托勒密不等式與面積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Express the area using the two diagonals and their included angle. Then apply Ptolemy's inequality to the product of the diagonals.

Olympiad Knowledge Points 奧數知識點

- **Euclidean Geometry 歐氏幾何**
- **Ptolemy Inequality 托勒密不等式**
- **Diagonal Area Formula 對角線面積公式**

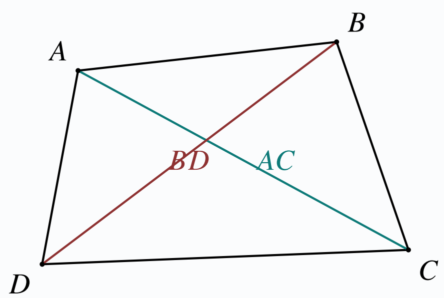

Figure 16. The area is controlled by the product of the diagonals.

Solution 解答

Let θ be the angle between the diagonals. Splitting the quadrilateral into four triangles around the intersection of the diagonals gives

$$2S = AC \cdot BD \sin\theta.$$

Hence

$$2S \le AC \cdot BD.$$

Ptolemy's inequality states that

$$AC \cdot BD \le AB \cdot CD + BC \cdot AD.$$

For completeness, identify the plane with the complex plane and let the vertices have complex coordinates a, b, c, d. The identity

$$(a - c)(b - d) = (a - b)(c - d) + (a - d)(b - c)$$

and the triangle inequality give precisely

$$|a-c|\,|b-d| \le |a-b|\,|c-d| + |a-d|\,|b-c|.$$

Combining the two inequalities yields

$$2S \le AB \cdot CD + BC \cdot AD.$$

□

Final Answer 最終答案. The required inequality is proved.

2021 | Senior Division高中組

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本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Finite differences and factorials 丨有限差分與階乘

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Rewrite the numerator as $n(n-1)+2n+1$. After introducing $a_n = (-1)^n/n!$, each summand becomes a second finite difference and the sum telescopes.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Telescoping Sums 望遠鏡和 | Finite Differences 有限差分 | Factorials 階乘

Solution 解答

For $n \ge 2$,

$$\frac{n^2+n+1}{n!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!} + \frac{1}{n!}.$$

Let

$$a_n = \frac{(-1)^n}{n!}.$$

Then

$$(-1)^n \frac{n^2+n+1}{n!} = a_{n-2} - 2a_{n-1} + a_n.$$

The initial part is

$$1 - 3 = -2.$$

Therefore, with $N = 2021$,

$$\begin{aligned} 1 + \sum_{n=1}^{N} (-1)^n \frac{n^2+n+1}{n!} &= -2 + \sum_{n=2}^{N} (a_{n-2} - 2a_{n-1} + a_n) \\ &= -2 + (a_N - a_{N-1}) - (a_1 - a_0) \\ &= a_N - a_{N-1}. \end{aligned}$$

Since $N = 2021$ is odd,

$$a_{2021} = -\frac{1}{2021!}, \qquad a_{2020} = \frac{1}{2020!}.$$

Hence the value is

$$-\frac{1}{2021!} - \frac{1}{2020!} = -\frac{2022}{2021!}.$$

Final Answer 最終答案. $-\dfrac{2022}{2021!}$.

Question 2 題目 2 The range of quadratic roots | 二次方程根的範圍

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Treat x as a proposed root. The equation requires $b = -x^2 - ax$. For fixed x, vary a over its interval and ask whether the resulting interval for $x^2 + ax$ meets $[-2, 2]$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Parameter Range 參數範圍 | Quadratic Equations 二次方程 | Interval Analysis 區間分析

Solution 解答

A real number x is a possible root if and only if there exists $a \in [-2, 2]$ such that

$$b = -x^2 - ax \in [-2, 2].$$

Equivalently, we require

$$|x^2 + ax| \leq 2$$

for some $a \in [-2, 2]$.
For fixed x, the term ax ranges over

$$[-2|x|, 2|x|].$$

Thus the smallest possible value of $|x^2 + ax|$ is

$$\max\{x^2 - 2|x|, 0\}.$$

A suitable a exists precisely when

$$x^2 - 2|x| \leq 2.$$

Putting $t = |x|$, this becomes

$$t^2 - 2t - 2 \leq 0,$$

so

$$0 \leq t \leq 1 + \sqrt{3}.$$

Therefore the complete range is

$$-(1 + \sqrt{3}) \leq x \leq 1 + \sqrt{3}.$$

At the endpoints one may take $a = -2\,\mathrm{sgn}(x)$ and $b = -2$, so the endpoints are included.

Final Answer 最終答案. $\boxed{[-1 - \sqrt{3}, 1 + \sqrt{3}]}$.

Question 3 題目 3 Distance from lattice points to a line | 格點到直線的距離

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the point-to-line distance formula. The numerator is a nonzero integer, and Bézout's identity shows that the smallest possible absolute value is attained.

Olympiad Knowledge Points 奧數知識點

Analytic Geometry 解析幾何 **Lattice Points 格點** **Bézout Identity 貝祖等式** **Distance Formula 距離公式**

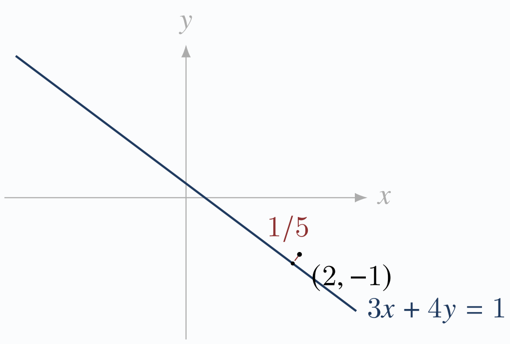

Figure 17. The lattice point $(2, -1)$ has distance $1/5$ from the line.

Solution 解答

The distance is

$$f(a, b) = \frac{|3a + 4b - 1|}{\sqrt{3^2 + 4^2}} = \frac{|3a + 4b - 1|}{5}.$$

Because a, b are integers, $3a + 4b - 1$ is an integer. Since the point is not on the line, this integer is nonzero, so

$$f(a, b) \geq \frac{1}{5}.$$

The bound is attainable: for $(a, b) = (2, -1)$,

$$3a + 4b - 1 = 6 - 4 - 1 = 1.$$

Therefore

$$f(2, -1) = \frac{1}{5}.$$

Final Answer 最終答案. $\min f(a, b) = \frac{1}{5}$.

Question 4 題目 4 Triangles in a triangular lattice | 三角形網格計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Count upward and downward triangles separately. For each fixed side length, the possible positions form a smaller triangular array. Sum these counts and simplify by parity of n.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 | Geometric Counting 幾何計數 | Summation 求和 | Parity Cases 奇偶分類

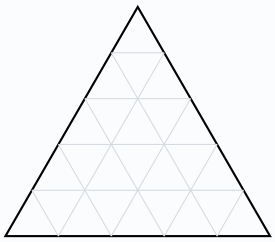

Figure 18. A triangular lattice of side length $n = 5$.

Solution 解答

An upward-pointing triangle of side length k can be placed in

$$\frac{(n-k+1)(n-k+2)}{2}$$

ways. Hence the total number of upward triangles is

$$U_n = \sum_{k=1}^{n} \frac{(n-k+1)(n-k+2)}{2} = \frac{n(n+1)(n+2)}{6}.$$

A downward-pointing triangle of side length k exists only when $2k \le n$, and its number of positions is

$$\frac{(n-2k+1)(n-2k+2)}{2}.$$

Thus

$$D_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(n-2k+1)(n-2k+2)}{2}.$$

If $n = 2m$, simplification gives

$$N_{2m} = U_{2m} + D_{2m} = \frac{m(m+1)(4m+1)}{2}.$$

If $n = 2m + 1$, it gives

$$N_{2m+1} = \frac{(m+1)(4m^2+7m+2)}{2}.$$

Both cases are represented compactly by

$$N_n = \left\lfloor \frac{n(n+2)(2n+1)}{8} \right\rfloor.$$

Final Answer 最終答案. $N_n = \left\lfloor \dfrac{n(n+2)(2n+1)}{8} \right\rfloor.$

Question 5 題目 5 Cyclic and homogeneous inequalities 丨循環與齊次不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

For part (i), take logarithms and apply the log-sum inequality to a cyclic permutation. For part (ii), manufacture each term on the right by a four-term AM–GM inequality and add the three results.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 **Log-Sum Inequality 對數和不等式** **Weighted AM–GM 加權算術幾何平均** **Homogeneity 齊次性**

Solution 解答

(i) Taking logarithms, it is enough to prove

$$(x-y)\ln x + (y-z)\ln y + (z-x)\ln z \geq 0.$$

The left-hand side can be written as

$$x\ln\frac{x}{z} + y\ln\frac{y}{x} + z\ln\frac{z}{y}.$$

By the log-sum inequality,

$$\begin{aligned} x\ln\frac{x}{z} + y\ln\frac{y}{x} + z\ln\frac{z}{y} &\geq (x+y+z)\ln\frac{x+y+z}{z+x+y} \\ &= 0. \end{aligned}$$

Exponentiating proves the result.
(ii) By AM–GM,

$$2a^4 + b^4 + c^4 \geq 4a^2bc.$$

Cyclically,

$$a^4 + 2b^4 + c^4 \geq 4ab^2c,$$

$$a^4 + b^4 + 2c^4 \geq 4abc^2.$$

Adding these three inequalities gives

$$4(a^4 + b^4 + c^4) \geq 4(a^2bc + ab^2c + abc^2),$$

which is the desired inequality.

Final Answer 最終答案. Both inequalities hold; equality in each occurs when $x = y = z$ or $a = b = c$, respectively.

Question 6 題目 6 Functional pairing ｜函數配對

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Pair the terms with arguments x and $1 - x$. Their sum is constant.

Olympiad Knowledge Points 奧數知識點

Functional Equations 函數關係 | Pairing 配對法 | Symmetry 對稱性

Solution 解答

For every real x,

$$f(1-x) = \frac{9^{1-x}}{9^{1-x}+3} = \frac{9}{9+3\cdot 9^x} = \frac{3}{3+9^x}.$$

Therefore

$$f(x) + f(1-x) = \frac{9^x}{9^x+3} + \frac{3}{9^x+3} = 1.$$

The terms pair as

$$\frac{k}{2021} \quad \text{and} \quad \frac{2021-k}{2021}.$$

There are $2020/2 = 1010$ pairs, each with sum 1. Hence the required sum is

$$1010.$$

Final Answer 最終答案. $1010.$

Question 7 題目 7 Average divisor counts | 平均因數個數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Interpret the floor sum as the total number of divisors of all integers up to n. The increment is therefore the divisor-count function $\tau(n+1)$. Compare this increment with the current average, using powers of 2 for increases and primes for decreases.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Divisor Function 因數個數函數** **Double Counting 雙重計數** **Harmonic Bounds 調和和估計**

Solution 解答

Let

$$S_n = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor.$$

The quantity S_n counts pairs (k, m) with $km \le n$, and therefore

$$S_n = \sum_{r=1}^{n} \tau(r),$$

where $\tau(r)$ is the number of positive divisors of r. Hence

$$S_{n+1} - S_n = \tau(n+1).$$

Since $a_n = S_n/n$,

$$a_{n+1} - a_n = \frac{\tau(n+1) - a_n}{n+1}.$$

(a) Take

$$n = 2^m - 1.$$

Then

$$\tau(n+1) = \tau(2^m) = m + 1.$$

Also,

$$a_n = \frac{1}{n} \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \le \sum_{k=1}^{n} \frac{1}{k} = H_n.$$

The standard bound $H_n < 1 + \ln n$ gives

$$a_n < 1 + m \ln 2 < m + 1.$$

Thus $\tau(n+1) > a_n$, so $a_{n+1} > a_n$. This holds for every $m \ge 1$, giving infinitely many such n.
(b) Yes. Let $p > 37$ be prime and take $n = p - 1$. Then

$$\tau(n+1) = \tau(p) = 2.$$

For $n > 36$,

$$a_n \geq \frac{1}{n}\left(n + \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor\right) \geq \frac{25}{12} - \frac{3}{n} > 2.$$

Therefore $\tau(n+1) < a_n$, and hence $a_{n+1} < a_n$. Since there are infinitely many primes, there are infinitely many such n.

Final Answer 最終答案. Both strict increases and strict decreases occur infinitely often.

Question 8 題目 8 A hidden angle bisector | 隱藏角平分線

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use coordinates adapted to the altitude: put D at the origin and AD on the vertical axis. Compute the intersections E and F; their slopes from D are negatives of each other, which means that DH bisects $\angle EDF$.

Olympiad Knowledge Points 奧數知識點

Analytic Geometry 解析幾何 | Angle Bisectors 角平分線 | Line Intersections 直線交點 | Symmetric Slopes 對稱斜率

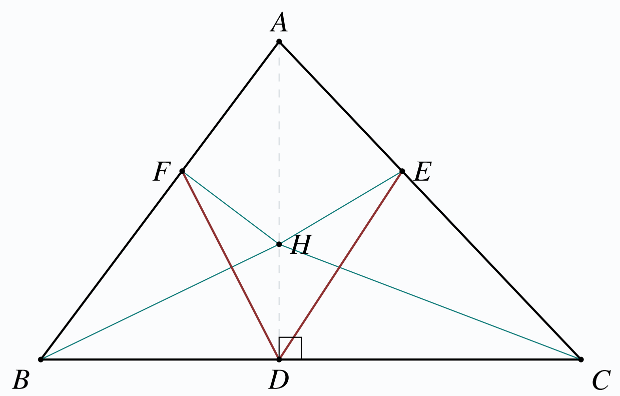

Figure 19. DE and DF are symmetric in direction about the altitude DH.

Solution 解答

Set

$$D = (0,0), \quad A = (0,a), \quad B = (-b,0), \quad C = (c,0), \quad H = (0,h),$$

where $a, b, c > 0$ and $0 < h < a$.
Solving the equations of lines BH and AC gives

$$E = \left(\frac{bc(a-h)}{ab+ch}, \frac{ah(b+c)}{ab+ch}\right).$$

Similarly, solving the equations of CH and AB gives

$$F = \left(-\frac{bc(a-h)}{ac+bh}, \frac{ah(b+c)}{ac+bh}\right).$$

Consequently,

$$\frac{x_E}{y_E} = \frac{bc(a-h)}{ah(b+c)},$$

while

$$\frac{x_F}{y_F} = -\frac{bc(a-h)}{ah(b+c)}.$$

Thus the rays DE and DF make equal angles with the positive vertical axis DH, on opposite sides. Therefore

$$\angle EDH = \angle FDH.$$

□

Final Answer 最終答案. DH bisects $\angle EDF$.

Question 9 題目 9 A polynomial difference operator | 多項式差分算子

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Study the leading term under the given operator. Degrees above 2 cannot disappear. Then substitute a general quadratic and impose cancellation of its linear term.

Olympiad Knowledge Points 奧數知識點

Polynomials 多項式 **Leading Coefficients 首項係數** **Finite-Difference Operators 差分算子**

Solution 解答

Suppose P has degree m and leading coefficient $c \neq 0$. For the monomial x^m,

$$(x+1)(x-1)^m - (x-1)x^m = (2-m)x^m + \text{lower-degree terms}.$$

If $m > 2$, the coefficient $(2-m)c$ is nonzero, so the result cannot be constant. Thus $\deg P \leq 2$.

Write

$$P(x) = Ax^2 + Bx + C.$$

Direct calculation gives

$$\begin{aligned}(x+1)P(x-1) - (x-1)P(x) \\ = (B-A)x + (A-B+2C).\end{aligned}$$

This is constant if and only if

$$B = A.$$

Therefore all solutions are

$$P(x) = A(x^2 + x) + C, \qquad A, C \in \mathbb{R}.$$

For these polynomials, the resulting constant is $2C$.

Final Answer 最終答案. $\boxed{P(x) = A(x^2 + x) + C}$ for arbitrary real constants A, C.

Question 10 題目 10 Tangents, areas, and a right angle | 切線、面積與直角

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Let the triangle angles be α, β, γ. Convert the two area equalities into trigonometric equations. Tangent–chord angles and the sine rule eliminate all lengths, leaving an equation that forces $\cos\gamma = 0$.

Olympiad Knowledge Points 奧數知識點

Euclidean Geometry 歐氏幾何 **Tangent–Chord Theorem 切線弦定理**
Area Ratios 面積比 **Sine Rule 正弦定理**

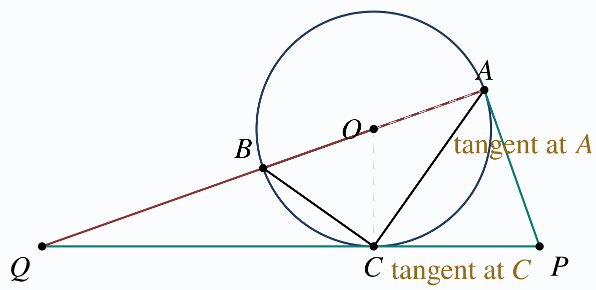

Figure 20. An exact representative of the final configuration: AB is a diameter once $\angle C = 90^\circ$.

Solution 解答

Let

$$\alpha = \angle A, \qquad \beta = \angle B, \qquad \gamma = \angle C.$$

Write the side lengths opposite these angles as a, b, c, and let the circumradius be R.
By the tangent–chord theorem,

$$\angle PAC = \angle PCA = \beta.$$

Thus triangle APC is isosceles, and

$$AC = 2\,PC\cos\beta, \qquad PC = \frac{b}{2\cos\beta}.$$

The equality $[ACP] = [ABC]$, using base AC, gives

$$PC\sin\beta = a\sin\gamma.$$

By the sine rule $a/b = \sin\alpha/\sin\beta$, so

$$\frac{b\sin\beta}{2\cos\beta} = a\sin\gamma$$

becomes

$$\sin^2\beta = 2\sin\alpha\sin\gamma\cos\beta. \tag{1}$$

Now consider triangle BQC. Since Q lies beyond B on line AB, its angles at B and C are

$$\angle QBC = 180^\circ - \beta, \qquad \angle BCQ = \alpha.$$

Hence

$$\angle BQC = \beta - \alpha.$$

By the sine rule in triangle BQC,

$$CQ = \frac{a\sin\beta}{\sin(\beta-\alpha)}.$$

The equality $[BQC] = [ABC]$ gives

$$\frac{1}{2}a\,CQ\sin\alpha = \frac{1}{2}ab\sin\gamma.$$

Substituting CQ and $a/b = \sin\alpha/\sin\beta$ yields

$$\sin^2\alpha = \sin\gamma\sin(\beta-\alpha).$$

Since $\gamma = 180^\circ - \alpha - \beta$,

$$\sin\gamma\sin(\beta-\alpha) = \sin(\alpha+\beta)\sin(\beta-\alpha) = \sin^2\beta - \sin^2\alpha.$$

Therefore

$$\sin^2\beta = 2\sin^2\alpha. \tag{2}$$

Combining (1) and (2),

$$\sin\alpha = \sin\gamma\cos\beta.$$

But

$$\sin\alpha = \sin(\beta+\gamma) = \sin\beta\cos\gamma + \cos\beta\sin\gamma.$$

Thus

$$\sin\beta\cos\gamma = 0.$$

Since $0 < \beta < 90^\circ$, $\sin\beta > 0$, so

$$\cos\gamma = 0.$$

Hence

$$\gamma = 90^\circ.$$

□

Final Answer 最終答案. $\boxed{\angle BCA = 90^\circ}$.

2022 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Telescoping radicals | 根式望遠鏡和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Rationalise each denominator. Every term becomes the difference of two consecutive square roots, so the sum telescopes.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Rationalisation 分母有理化 | Telescoping Sum 望遠鏡求和

Solution 解答

For $1 \le k \le 2021$,

$$\frac{1}{\sqrt{k}+\sqrt{k+1}} = \frac{\sqrt{k+1}-\sqrt{k}}{(k+1)-k} = \sqrt{k+1}-\sqrt{k}.$$

Therefore

$$\begin{aligned} S &= \sum_{k=1}^{2021}\left(\sqrt{k+1}-\sqrt{k}\right) \\ &= \sqrt{2022}-1. \end{aligned}$$

Final Answer 最終答案. $\sqrt{2022}-1$, so the correct choice is **A.**

Question 2 題目 2 Substitution | 代換

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Treat the repeated expression $\sqrt{x}+\sqrt{y}$ as a single positive variable.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Substitution 代換 | Quadratic Equations 二次方程

Solution 解答

Set
$$t = \sqrt{x} + \sqrt{y} > 0.$$
Then
$$t(t-1) = 6,$$
so
$$t^2 - t - 6 = 0, \qquad (t-3)(t+2) = 0.$$
Since $t > 0$, we obtain $t = 3$.

Final Answer 最終答案. 3, so the correct choice is **C**.

Question 3 題目 3 Symmetric expressions ︱對稱式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Factor the target expression before substituting the given elementary symmetric quantities.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Factorisation 因式分解 | Symmetric Expressions 對稱式

Solution 解答

We have
$$a^2b + ab^2 = ab(a+b).$$
Hence
$$a^2b + ab^2 = (-2022)(2) = -4044.$$

Final Answer 最終答案. -4044, so the correct choice is **A**.

Question 4 題目 4 Partial fractions ︱部分分式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Clear denominators and compare the coefficients of x and the constant term.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Partial Fractions 部分分式 | Coefficient Comparison 比較係數

Solution 解答

Since $x^2 - 1 = (x-1)(x+1)$,

$$x - 3 = A(x-1) + B(x+1).$$

Thus

$$A + B = 1, \qquad -A + B = -3.$$

Solving gives

$$A = 2, \qquad B = -1.$$

Therefore

$$A - B = 2 - (-1) = 3.$$

Final Answer 最終答案. 3, so the correct choice is **C**.

Question 5 題目 5 Solution sets of inequalities | 不等式解集

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Rewrite each inequality in the form of a bound on x and match the two endpoints of the stated interval.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Linear Inequalities 一次不等式 | Interval Matching 區間配對

Solution 解答

The inequalities are equivalent to

$$x > a + 1, \qquad x < b - 1.$$

Hence their common solution set is

$$a + 1 < x < b - 1.$$

Comparing with $1 < x < 3$ gives

$$a + 1 = 1, \qquad b - 1 = 3.$$

Thus

$$a = 0, \qquad b = 4,$$

and consequently

$$a^2b^2 = 0.$$

Final Answer 最終答案. 0, so the correct choice is **A.**

Question 6 題目 6 Integer-valued rational expressions | 整值分式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use polynomial division. The problem then becomes a divisor-counting problem for a fixed integer.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | **Divisibility 整除** | **Polynomial Division 多項式除法**

Solution 解答

We have

$$n^2 + 7 = (n + 3)(n - 3) + 16.$$

Therefore

$$\frac{n^2 + 7}{n + 3} = n - 3 + \frac{16}{n + 3}.$$

This is an integer exactly when $n + 3$ is a nonzero integer divisor of 16. The integer divisors of 16 are

$$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16,$$

so there are 10 possible values of $n + 3$, and hence 10 possible integers n.

Final Answer 最終答案. 10, so the correct choice is **D.**

Question 7 題目 7 Fractional parts | 小數部分

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The fractional parts repeat with period 5. Count complete periods and then handle the final four terms.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Fractional Parts 小數部分** **Periodicity 週期性**

Solution 解答

For consecutive integers, the fractional parts of division by 5 cycle as

$$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 0,$$

whose sum is 2. There are

$$4044 - 2021 + 1 = 2024 = 404 \cdot 5 + 4$$

terms. Since $2021 \equiv 1 \pmod 5$, the last four residual terms have fractional parts

$$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5},$$

which also sum to 2. Therefore

$$S = 404 \cdot 2 + 2 = 810.$$

Final Answer 最終答案. 810.

Question 8 題目 8 Factorisation | 因式分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Group the expression as a quadratic in x, or inspect it as a product of two linear forms.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Factorisation 因式分解** **Bilinear Forms 雙線性式**

Solution 解答

Direct expansion shows that

$$(x-2y)(x+y-2) = x^2 - xy - 2x - 2y^2 + 4y.$$

Hence

$$x^2 - 2x - 2y^2 + 4y - xy = (x-2y)(x+y-2).$$

Final Answer 最終答案. $(x-2y)(x+y-2)$.

Question 9 題目 9 Sign of a rational expression | 分式符號

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Replace the consecutive gaps by positive variables. The sign then becomes transparent.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Sign Analysis 符號分析** **Gap Substitution 差量代換**

Solution 解答

Let

$$x = b - a > 0, \qquad y = c - b > 0.$$

Then $c - a = x + y$, and

$$\frac{1}{a-b} + \frac{1}{b-c} + \frac{1}{c-a} = -\frac{1}{x} - \frac{1}{y} + \frac{1}{x+y}.$$

Since

$$\frac{1}{x+y} < \frac{1}{x}$$

and $1/y > 0$, the expression is strictly negative. More explicitly,

$$-\frac{1}{x} - \frac{1}{y} + \frac{1}{x+y} = -\frac{x^2+xy+y^2}{xy(x+y)} < 0.$$

Final Answer 最終答案. The expression is strictly negative.

Question 10 題目 10 Parity counting | 奇偶分類計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First count all triples with even sum by parity type, then subtract the few triples whose sum is below 10.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 | Parity 奇偶性 | Complementary Counting 補集計數

Solution 解答

An even sum occurs in exactly two cases:

- three even digits;
- one even digit and two odd digits.

There are five even digits and five odd digits, so the total number of even-sum triples is

$$\binom{5}{3}+\binom{5}{1}\binom{5}{2}=10+50=60.$$

The even-sum triples whose sum is less than 10 are

$$\{0,1,3\},\{0,1,5\},\{0,1,7\},\{0,2,4\},\{0,2,6\},\{0,3,5\},$$
$$\{1,2,3\},\{1,2,5\},\{1,3,4\}.$$

There are 9 of them. Hence the required number is

$$60-9=51.$$

Final Answer 最終答案. 51.

Question 11 題目 11 Area ratios in a parallelogram | 平行四邊形面積比

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use vectors with A as the origin. The intersection point F is obtained by equating two affine parametrisations, after which the area ratio follows from a determinant.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Vectors 向量** **Area Ratios 面積比**

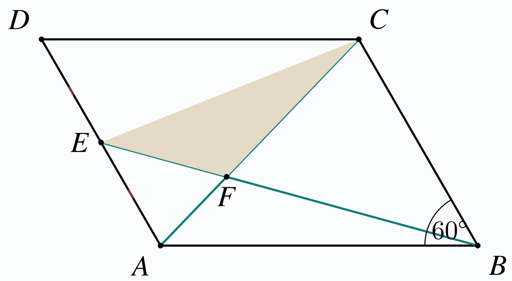

Figure 21. The given angle is $\angle B = 60^\circ$, and F satisfies $AF : FC = 1 : 2$.

Solution 解答

Let
$$\overrightarrow{AB} = \mathrm{u}, \qquad \overrightarrow{AD} = \mathrm{v},$$
and put A at the origin. Then
$$B = \mathrm{u}, \quad D = \mathrm{v}, \quad C = \mathrm{u} + \mathrm{v}, \quad E = \frac{1}{2}\mathrm{v}.$$
Since $F \in AC$, write
$$F = t(\mathrm{u} + \mathrm{v}).$$
Since $F \in BE$, write
$$F = \mathrm{u} + s\left(\frac{1}{2}\mathrm{v} - \mathrm{u}\right) = (1 - s)\mathrm{u} + \frac{s}{2}\mathrm{v}.$$
Comparing coefficients gives
$$t = 1 - s, \qquad t = \frac{s}{2},$$
so $s = 2/3$ and $t = 1/3$. Thus
$$F = \frac{1}{3}(\mathrm{u} + \mathrm{v}).$$
Now
$$\overrightarrow{EF} = \frac{1}{3}\mathrm{u} - \frac{1}{6}\mathrm{v}, \qquad \overrightarrow{EC} = \mathrm{u} + \frac{1}{2}\mathrm{v}.$$
Therefore
$$S_{EFC} = \frac{1}{2}\left|\det\left(\frac{1}{3}\mathrm{u} - \frac{1}{6}\mathrm{v}, \mathrm{u} + \frac{1}{2}\mathrm{v}\right)\right| = \frac{1}{6}|\det(\mathrm{u}, \mathrm{v})|.$$
Since $S_{ABCD} = |\det(\mathrm{u}, \mathrm{v})|$, we obtain
$$\frac{S_{EFC}}{S_{ABCD}} = \frac{1}{6}.$$

Final Answer 最終答案. $\frac{1}{6}$.

Question 12 題目 12 Angle trisectors, median and altitude | 三等分線、中線與高

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let $\angle A = 3\theta$. The order condition $\angle B < \angle C$ identifies which trisector is the altitude. Then apply the trigonometric form of the cevian ratio to the median.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Trigonometric Ceva 三角形截線比 | Median and Altitude 中線與高 | Angle Trisection 角三等分

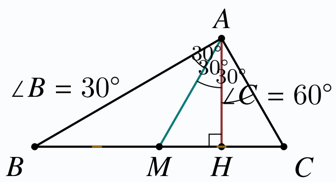

Figure 22. The extremal geometry forced by the conditions: AM and AH are the two trisectors of the right angle at A.

Solution 解答

Write
$$\angle A = 3\theta.$$
Let the trisectors from the side AB be the rays making angles θ and 2θ with AB. If the first trisector were the altitude, then
$$\angle B = 90^\circ - \theta, \qquad \angle C = 90^\circ - 2\theta,$$
which would give $\angle B > \angle C$, contrary to the hypothesis. Hence the second trisector is the altitude and the first is the median. Therefore
$$\angle B = 90^\circ - 2\theta, \qquad \angle C = 90^\circ - \theta.$$
Let AM be the median. By the trigonometric cevian ratio,
$$\frac{BM}{MC} = \frac{AB \sin \angle BAM}{AC \sin \angle MAC} = \frac{c \sin \theta}{b \sin 2\theta}.$$
Since $BM = MC$,
$$\frac{c}{b} = \frac{\sin 2\theta}{\sin \theta} = 2\cos\theta.$$

On the other hand, the sine rule gives

$$\frac{c}{b} = \frac{\sin C}{\sin B} = \frac{\cos\theta}{\cos 2\theta}.$$

Thus

$$2\cos\theta = \frac{\cos\theta}{\cos 2\theta}.$$

Since $0 < \theta < 45^\circ$, $\cos\theta > 0$, and therefore

$$\cos 2\theta = \frac{1}{2}.$$

Hence $2\theta = 60^\circ$, so $\theta = 30^\circ$. Consequently

$$\angle C = 90^\circ - \theta = 60^\circ.$$

Final Answer 最終答案. $\angle C = 60^\circ$.

Question 13 題目 13 A factor of a Mersenne-type number | 梅森型數的因數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Look for a small modulus for which the multiplicative order of 2 divides 2022.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | **Modular Arithmetic 模算術** | **Multiplicative Order 乘法階**

Solution 解答

Since

$$2^6 = 64 \equiv 1 \pmod{63}$$

and

$$2022 = 6 \cdot 337,$$

we have

$$2^{2022} = (2^6)^{337} \equiv 1^{337} \equiv 1 \pmod{63}.$$

Therefore

$$63 \mid 2^{2022} - 1.$$

Also $10 < 63 < 99$, so $d = 63$ is valid.

Final Answer 最終答案. One possible divisor is $d = 63$.

Question 14 題目 14 Intersections determined by five points 丨五點連線交點

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Editorial Note 編按

As printed, the number is not uniquely determined. The condition that no three of the original five points are collinear does not force all five points to be in convex position, nor does it prevent different segment intersections from coinciding. Two valid configurations are shown below.

Solution Strategy 思路解析

Test whether the stated hypotheses determine a unique combinatorial configuration. A convex configuration and a configuration with one interior point give different answers.

Olympiad Knowledge Points 奧數知識點

Combinatorial Geometry 組合幾何 **Convexity 凸性** **Counterexample 反例**

Convex configuration: 5 crossings One interior point: 3 crossings

Figure 23. Both configurations satisfy the printed condition, but their numbers of crossings differ.

Solution 解答

If the five points are in convex position and in general position, every choice of four vertices determines exactly one crossing of the two diagonals of that quadrilateral. Hence the number of crossings is

$$\binom{5}{4} = 5.$$

However, take four points forming a quadrilateral and a fifth point strictly inside it, chosen so that it lies on none of the diagonals. The complete set of ten joining segments can then have only 3 proper crossings, as in the diagram. Thus the printed hypotheses admit at least two different answers, 5 and 3.

Final Answer 最終答案. The number is **not uniquely determined** by the stated conditions. If convex position was intended, the answer is 5.

Question 15 題目 15 Integer optimisation with congruences | 同餘限制下的整數最值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Set $x + y = 3m$. The congruence modulo 5 then restricts y, while the objective can be expressed in terms of m and the quantity $2x + y$.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Linear Congruences 線性同餘 | Integer Optimisation 整數最值

Solution 解答

Since $3 \mid x + y$, write

$$x + y = 3m$$

for a nonnegative integer m. The condition $5 \mid x + 2y$ becomes

$$3m + y \equiv 0 \pmod 5,$$

so

$$y \equiv 2m \pmod 5.$$

Also

$$7x + 5y = 2(2x + y) + 3(x + y) = 2(2x + y) + 9m.$$

Because $2x + y \geq 99$,

$$7x + 5y \geq 198 + 9m.$$

Moreover,

$$2x + y \leq 2(x + y) = 6m,$$

so $6m \geq 99$, hence $m \geq 17$.
If $m = 17$, then $x + y = 51$ and $y \equiv 34 \equiv 4 \pmod 5$. But

$$2x + y = 102 - y \geq 99$$

would require $y \leq 3$, impossible.
For $m = 18$, we have $x + y = 54$ and $y \equiv 36 \equiv 1 \pmod 5$. The condition

$$2x + y = 108 - y \geq 99$$

requires $y \leq 9$, so the possible values are $y = 1$ or $y = 6$. Taking $y = 6$ gives

$$x = 48, \qquad 2x + y = 102,$$

and

$$7x + 5y = 7 \cdot 48 + 5 \cdot 6 = 366.$$

For $m \geq 19$,

$$7x + 5y \geq 198 + 9 \cdot 19 = 369 > 366.$$

Therefore 366 is minimal.

Final Answer 最終答案. 366, attained at $(x, y) = (48, 6)$.

Question 16 題目 16 Maximum area with fixed circumradius | 定外接圓半徑的最大面積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Express the area in terms of the circumradius and the three angles, then maximise the product $\sin A \sin B \sin C$ under $A + B + C = 180°$.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Circumradius 外接圓半徑 | Trigonometric Optimisation 三角最值

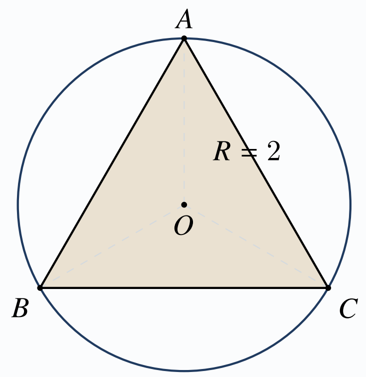

Figure 24. The maximum is attained by the equilateral triangle.

Solution 解答

Using $a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$, the area is

$$K = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C.$$

With $R = 2$,

$$K = 8 \sin A \sin B \sin C.$$

For $A + B + C = 180°$, the product $\sin A \sin B \sin C$ is maximised at

$$A = B = C = 60°.$$

For example, this follows from the concavity of $\log \sin x$ on $(0, \pi)$ and Jensen's inequality. Hence

$$\sin A \sin B \sin C \le (\sin 60°)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

Therefore
$$K \le 8 \cdot \frac{3\sqrt{3}}{8} = 3\sqrt{3}.$$
Equality holds for an equilateral triangle.

Final Answer 最終答案. $3\sqrt{3}$.

2022 | Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Maximum area with fixed circumradius 丨定外接圓半徑的最大面積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Write the area as $2R^2 \sin A \sin B \sin C$, and maximise the sine product under $A + B + C = \pi$.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Circumradius 外接圓半徑 | Trigonometric Optimisation 三角最值

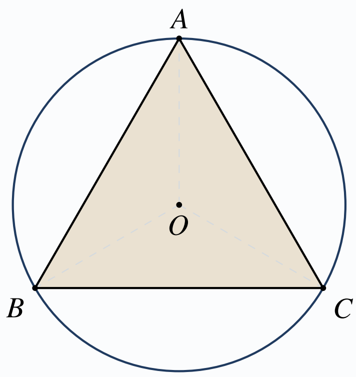


Figure 25. The extremal triangle is equilateral.

Solution 解答

For a triangle with circumradius R,

$$K = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C.$$

When $R = 2$,

$$K = 8 \sin A \sin B \sin C.$$

Since $A + B + C = \pi$ and $\log \sin x$ is concave on $(0, \pi)$,

$$\sin A \sin B \sin C \le \left(\sin \frac{A+B+C}{3}\right)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

Thus

$$K \le 3\sqrt{3},$$

with equality exactly when $A = B = C = 60^\circ$.

Final Answer 最終答案. $3\sqrt{3}$.

Question 2 題目 2 Parity counting 丨奇偶分類計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Count all triples of even total by parity type, and subtract those with sum below 10.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 | **Parity 奇偶性** | **Complementary Counting 補集計數**

Solution 解答

There are five even digits and five odd digits. An even sum is obtained from either three even digits or one even and two odd digits. Hence there are

$$\binom{5}{3} + \binom{5}{1}\binom{5}{2} = 10 + 50 = 60$$

even-sum triples in total. The triples with even sum less than 10 are

$$\{0,1,3\}, \{0,1,5\}, \{0,1,7\}, \{0,2,4\}, \{0,2,6\}, \{0,3,5\},$$
$$\{1,2,3\}, \{1,2,5\}, \{1,3,4\},$$

so there are 9 exclusions. Therefore the answer is

$$60 - 9 = 51.$$

Final Answer 最終答案. 51.

Question 3 題目 3 Minimum intercept area 丨截距三角形的最小面積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the intercept form $x/a + y/b = 1$. The point condition gives $3/a + 4/b = 1$, and AM–GM yields a lower bound for ab.

Olympiad Knowledge Points 奧數知識點

Analytic Geometry 解析幾何 | Intercept Form 截距式 | AM–GM 均值不等式

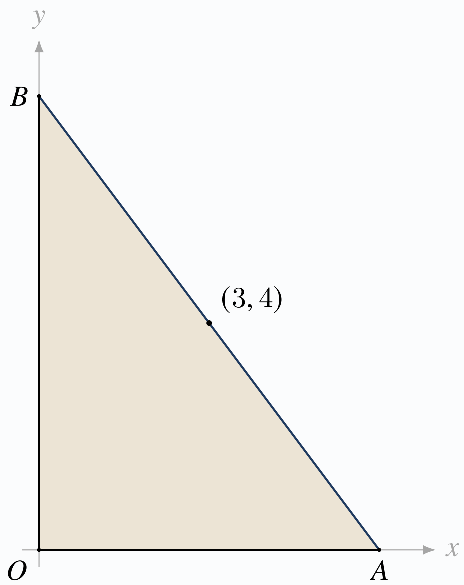


Figure 26. Equality occurs for intercepts $OA = 6$ and $OB = 8$.

Solution 解答

Let

$$OA = a, \qquad OB = b,$$

where $a, b > 0$. The line has equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Since $(3, 4)$ lies on it,

$$\frac{3}{a} + \frac{4}{b} = 1.$$

By AM–GM,

$$1 = \frac{3}{a} + \frac{4}{b} \geq 2\sqrt{\frac{12}{ab}}.$$

Therefore

$$ab \geq 48.$$

The area is

$$[OAB] = \frac{1}{2}ab \geq 24.$$

Equality in AM–GM requires

$$\frac{3}{a} = \frac{4}{b} = \frac{1}{2},$$

which gives $a = 6$ and $b = 8$. These intercepts indeed define a line through $(3, 4)$.

Final Answer 最終答案. 24.

Question 4 題目 4 A cubic minimum | 三次式最小值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Shift the target by the expected minimum and factor the resulting polynomial into a visibly nonnegative expression.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Polynomial Factorisation 多項式分解 | Optimisation 最值

Solution 解答

We have
$$f(x) + 3 = x^3 - 3x + 2 = (x-1)^2(x+2).$$
For $x > 0$, both factors on the right are nonnegative, so
$$f(x) + 3 \geq 0.$$
Hence
$$f(x) \geq -3.$$
Equality holds at $x = 1$.

Final Answer 最終答案. -3, attained at $x = 1$.

Question 5 題目 5 Induction for the sum of cubes | 立方和歸納證明

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use mathematical induction in its standard two-stage form. First verify the formula at $n = 1$. For the induction step, write $S_{n+1} = S_n + (n+1)^3$, substitute the induction hypothesis, factor out $(n+1)^2$, and simplify the remaining quadratic to $(n+2)^2/4$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Mathematical Induction 數學歸納法 | Sum of Cubes 立方和

Solution 解答

For $n = 1$,

$$S_1 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2.$$

Now assume

$$S_n = \left(\frac{n(n+1)}{2}\right)^2.$$

Then

$$\begin{aligned} S_{n+1} &= S_n + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4} + n + 1\right) \\ &= (n+1)^2\frac{(n+2)^2}{4} \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2. \end{aligned}$$

This completes the induction step.

Final Answer 最終答案. $S_n = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.

Question 6 題目 6 Intersections determined by five points ｜五點連線交點

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Editorial Note 編按

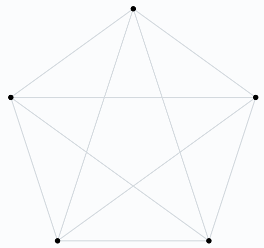
The printed conditions do not determine a unique answer. A convex set of five points gives 5 crossings, while a valid configuration with one point inside a quadrilateral can give 3 crossings.

Solution Strategy 思路解析

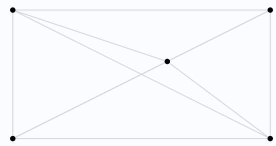
Check whether the hypotheses force a unique order type. Construct two admissible configurations with different crossing counts.

Olympiad Knowledge Points 奧數知識點

Combinatorial Geometry 組合幾何 | Convexity 凸性 | Counterexample 反例



Convex: 5 crossings



One interior point: 3 crossings

Figure 27. The number of crossings depends on the configuration.

Solution 解答

For five points in convex position, every choice of four points determines one pair of crossing diagonals. Thus a generic convex configuration has

$$\binom{5}{4} = 5$$

crossings. On the other hand, the second configuration in the figure satisfies the condition that no three original points are collinear, but has only 3 proper segment crossings. Therefore the number is not fixed by the stated hypotheses.

Final Answer 最終答案. The question is underdetermined as printed. If convex position was intended, the answer is 5.

Question 7 題目 7 A deletion game and pairing strategy | 刪數遊戲與配對策略

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

On the first move, remove the middle block $47, \ldots, 55$. The remaining numbers form 46 pairs whose members differ by 55. After every opponent move, restore the invariant that all surviving numbers occur in complete pairs.

Olympiad Knowledge Points 奧數知識點

Combinatorial Games 組合博弈 | Pairing Strategy 配對策略 | Invariant 不變量

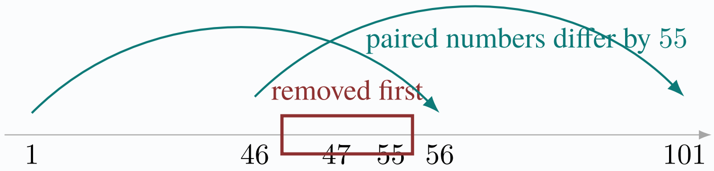


Figure 28. After the first move, pair i with $i + 55$ for $1 \le i \le 46$.

Solution 解答

On the first move, the first player removes

$$47, 48, \ldots, 55.$$

The remaining 92 numbers are partitioned into the 46 pairs

$$(1, 56), (2, 57), \ldots, (46, 101).$$

Each pair has difference 55.
Suppose before the second player's move the remaining set is a union of complete pairs. During that move, let the second player remove both members of d pairs and one member of each of s further pairs. Since exactly 9 numbers are removed,

$$2d + s = 9.$$

The first player removes the partners of the s singly removed numbers, using s removals, and then removes both members of any d intact pairs, using the remaining $2d$ removals. Thus exactly 9 numbers are removed, and the surviving set is again a union of complete pairs.
After the initial move there are five such response rounds. Each round eliminates exactly 9 complete pairs, so from 46 pairs exactly one pair remains. Its two members differ by 55.

Final Answer 最終答案. The first player can guarantee a score of 55.

Question 8 題目 8 A circular letter process | 圓周字母迭代

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Work backwards from a hypothetical first all-B state. Its predecessor must be constant, while an all-A state cannot have a predecessor on an odd cycle.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Invariant and Preimage 不變量與逆像** **Odd Cycles 奇環**

Solution 解答

Suppose all letters become B for the first time after some round. For a new letter to be B, its two neighbouring old letters must be equal. Hence, if every new letter is B, every pair of adjacent old letters is equal. Since the positions form one circle, all old letters must therefore be identical: either all A or all B.
By minimality, the preceding state cannot already be all B. It would have to be all A. However, to produce all A in one round, every pair of adjacent letters in the previous state would have to be different. This would require the letters to alternate $A, B, A, B, \dots$ around the circle. Such an alternating colouring is impossible on a circle with 99 positions, because 99 is odd.
This contradiction shows that an all-B state can never be reached.

Final Answer 最終答案. No, it is impossible.

Question 9 題目 9 A cyclic product inequality | 循環乘積不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Part (i) factors into a square. For part (ii), clear the common denominator and multiply the adjacent two-variable inequalities

$$(1+x_i^2)(1+x_{i+1}^2) \ge (1+x_i x_{i+1})^2.$$

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Factorisation 因式分解 | Cyclic Products 循環乘積

Solution 解答

For part (i), direct expansion gives

$$\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)-\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_1}\right) = \frac{(x_1-x_2)^2}{x_1x_2} \ge 0.$$

For part (ii), after multiplying both sides by the positive quantity $x_1x_2\cdots x_n$, it is enough to prove

$$\prod_{i=1}^{n}(1+x_i^2) \ge \prod_{i=1}^{n}(1+x_i x_{i+1}).$$

For each i,

$$(1+x_i^2)(1+x_{i+1}^2)-(1+x_i x_{i+1})^2 = (x_i-x_{i+1})^2 \ge 0.$$

Thus

$$(1+x_i^2)(1+x_{i+1}^2) \ge (1+x_i x_{i+1})^2.$$

Multiplying these n inequalities cyclically yields

$$\left(\prod_{i=1}^{n}(1+x_i^2)\right)^2 \geq \left(\prod_{i=1}^{n}(1+x_ix_{i+1})\right)^2.$$

Both products are positive, so taking square roots proves the result.

Final Answer 最終答案. Both inequalities hold; equality occurs when all relevant adjacent variables are equal.

Question 10 題目 10 Best constant in an inequality | 不等式最佳常數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Apply the Cauchy–Schwarz inequality in Engel form, then test the symmetric point $x_1 = \cdots = x_n = 1/n$ to prove sharpness.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | **Cauchy–Schwarz 柯西不等式** | **Best Constant 最佳常數**

Solution 解答

By Cauchy–Schwarz in Engel form,

$$\sum_{j=1}^{n}\frac{x_j^2}{1-x_j} \geq \frac{(x_1+\cdots+x_n)^2}{(1-x_1)+\cdots+(1-x_n)}.$$

Since $\sum x_j = 1$,

$$(1-x_1)+\cdots+(1-x_n) = n-1.$$

Therefore

$$\sum_{j=1}^{n}\frac{x_j^2}{1-x_j} \geq \frac{1}{n-1}.$$

Thus $C = n-1$ is sufficient.
If

$$x_1 = x_2 = \cdots = x_n = \frac{1}{n},$$

then

$$\sum_{j=1}^{n}\frac{x_j^2}{1-x_j} = n\cdot\frac{1/n^2}{1-1/n} = \frac{1}{n-1}.$$

Hence any admissible C must satisfy $C \geq n-1$.

Final Answer 最終答案. $C = n - 1$.

Question 11 題目 11 A three-variable product inequality | 三元乘積不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Introduce $p = \sqrt{xy}$, $q = \sqrt{yz}$ and $r = \sqrt{zx}$. The left side becomes a product of three simple linear expressions. AM–GM reduces the problem to a sum-of-squares inequality.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Radical Substitution 根式代換 | AM–GM 均值不等式 | Sum of Squares 平方和

Solution 解答

Set

$$p = \sqrt{xy}, \qquad q = \sqrt{yz}, \qquad r = \sqrt{zx}.$$

Then

$$pr = xq, \qquad pq = yr, \qquad qr = zp,$$

and $pqr = xyz$. Hence

$$xyz(x+2)(y+2)(z+2) = (pr+2q)(pq+2r)(qr+2p).$$

By AM–GM,

$$\sqrt[3]{xyz(x+2)(y+2)(z+2)} \le \frac{pq+pr+qr+2p+2q+2r}{3}.$$

Now

$$\begin{aligned}
&3 + 2(p^2+q^2+r^2) - (pq+pr+qr+2p+2q+2r)\\
&\quad = (p-1)^2 + (q-1)^2 + (r-1)^2 + \frac{1}{2}\left((p-q)^2 + (q-r)^2 + (r-p)^2\right)\\
&\quad \ge 0.
\end{aligned}$$

Therefore

$$pq+pr+qr+2p+2q+2r \le 3 + 2(p^2+q^2+r^2).$$

Since

$$p^2+q^2+r^2 = xy+yz+zx,$$

we obtain

$$\sqrt[3]{xyz(x+2)(y+2)(z+2)} \le 1 + \frac{2(xy+yz+zx)}{3}.$$

Cubing both sides proves the inequality.

Final Answer 最終答案. The stated inequality holds, with equality at $x = y = z = 1$.

Question 12 題目 12 Pairwise coprime denominators | 兩兩互素分母

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Put the expression over the common denominator abc. Integrality forces the divisibility conditions $a \mid b + c$, $b \mid c + a$, and $c \mid a + b$. Order the variables and exploit the largest one.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Pairwise Coprimality 兩兩互素 | Divisibility 整除 | Extremal Ordering 極值排序

Solution 解答

Let the expression be N. Multiplying by abc gives

$$Nabc = a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2.$$

Reducing modulo a yields

$$0 \equiv bc(b + c) \pmod{a}.$$

Since a is coprime to both b and c, it is coprime to bc, so

$$a \mid b + c.$$

Similarly,

$$b \mid c + a, \qquad c \mid a + b.$$

Because the expression is symmetric, assume

$$a \le b \le c.$$

Since $c \mid a + b$ and $a + b \le 2c$, either

$$a + b = c$$

or

$$a + b = 2c.$$

The second case forces $a = b = c$, and pairwise coprimality then gives

$$(a, b, c) = (1, 1, 1).$$

This gives $N = 6$.
Now suppose $a + b = c$. Since

$$b \mid a + c = 2a + b,$$

we have $b \mid 2a$. But $\gcd(a,b)=1$, so $b \mid 2$. As $a \le b$, the possibilities are

$$(a,b)=(1,1) \quad \text{or} \quad (a,b)=(1,2).$$

They give respectively

$$(a,b,c)=(1,1,2) \quad \text{and} \quad (1,2,3).$$

For $(1,1,2)$,

$$N = 1+2+2+1+\frac{1}{2}+\frac{1}{2}=7.$$

For $(1,2,3)$,

$$N = 2+3+\frac{3}{2}+\frac{1}{2}+\frac{1}{3}+\frac{2}{3}=8.$$

All permutations give the same values because the expression is symmetric.

Final Answer 最終答案. The possible positive integers are $\boxed{6,7,8}$.

2022/2023 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Divisor counting | 約數個數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Factorise 2023 into prime powers and apply the divisor-counting formula.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Prime Factorisation 質因數分解** **Divisor Function 約數函數**

Solution 解答

We have

$$2023 = 7 \cdot 17^2.$$

Hence the number of positive divisors is

$$(1 + 1)(2 + 1) = 6.$$

Final Answer 最終答案. 6, so the correct choice is **C**.

Question 2 題目 2 Signs of cyclic ratios | 循環比值的符號

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Use the product of the three ratios to determine the parity of the number of negative terms, then rule out the case where all three are positive.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Sign Analysis 符號分析** **Cyclic Expressions 循環式**

Solution 解答

Since a, b, c are distinct, all denominators are nonzero. Multiplying the three ratios gives

$$\frac{a-b}{b-c} \cdot \frac{b-c}{c-a} \cdot \frac{c-a}{a-b} = 1.$$

Therefore the number of negative factors is even: it is either 0 or 2.
If all three ratios were positive, then $a-b$, $b-c$, and $c-a$ would have the same sign. This is impossible because

$$(a-b)+(b-c)+(c-a)=0$$

and none of the three differences is zero. Hence exactly two ratios are negative.

Final Answer 最終答案. 2, so the correct choice is **C**.

Question 3 題目 3 Comparing large powers | 大冪比較

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Rewrite r as a power of 2 and compare the logarithms of the three quantities.

Olympiad Knowledge Points 奧數知識點

- Algebra 代數
- Exponential Comparison 指數比較
- Logarithmic Estimation 對數估值

Solution 解答

First,

$$r = 64^{12} = 2^{72}.$$

Since $3^5 = 243 > 2^7$, raising both sides to the 14th power gives

$$p = 3^{70} > 2^{98} > 2^{72} = r.$$

Also,

$$17^4 = 83521 > 59049 = 3^{10}.$$

Raising this inequality to the seventh power gives

$$q = 17^{28} > 3^{70} = p.$$

Therefore

$$r < p < q.$$

Final Answer 最終答案. $r < p < q$, so the correct choice is **D**.

Question 4 題目 4 Nearest perfect cube | 最接近的完全立方數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Locate 2023 between two consecutive cubes and compare the two distances.

Olympiad Knowledge Points 奧數知識點

Number Sense 數感 | Perfect Powers 完全冪 | Estimation 估值

Solution 解答

The consecutive cubes around 2023 are

$$12^3 = 1728, \qquad 13^3 = 2197.$$

Their distances from 2023 are

$$2023 - 1728 = 295, \qquad 2197 - 2023 = 174.$$

Hence 2197 is closer.

Final Answer 最終答案. 2197, so the correct choice is **D**.

Question 5 題目 5 Symmetric expressions | 對稱式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Express $x^2 + y^2$ and xy in terms of $(x + y)^2$ and $(x - y)^2$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Symmetric Expressions 對稱式 | Sum and Difference 和差變換

Solution 解答

Adding the two given equations,

$$2(x^2 + y^2) = 2025 + 289 = 2314,$$

so

$$x^2 + y^2 = 1157.$$

Subtracting them,

$$4xy = 2025 - 289 = 1736,$$

so $xy = 434$. Therefore

$$x^2 - xy + y^2 = 1157 - 434 = 723.$$

Final Answer 最終答案. 723.

Question 6 題目 6 Diophantine factorisation ｜丟番圖因式分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First bound the smallest variable x. Then reduce each remaining case to a factorisation of the form $(u - r)(v - r) = N$.

Olympiad Knowledge Points 奧數知識點

- Number Theory 數論
- Diophantine Equations 丟番圖方程
- Bounding and Factorisation 估界與因式分解

Solution 解答

If $x \geq 3$, then $y, z \geq 4$ and

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) < \left(\frac{4}{3}\right)^3 < 3,$$

a contradiction. Hence $x = 1$ or $x = 2$.
If $x = 1$, then

$$2\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) = 3.$$

After clearing denominators,

$$yz - 2y - 2z = 2,$$

so

$$(y - 2)(z - 2) = 6.$$

Because $1 < y < z$, the positive factor pairs of 6 give

$$(y,z) = (3,8), \qquad (4,5).$$

If $x = 2$, then

$$\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) = 2,$$

so

$$(y-1)(z-1) = 2.$$

The only positive factor pair gives $y = 2$, which violates $x < y$.
Thus there are exactly two triples.

Final Answer 最終答案. 2 triples: $(1,3,8)$ and $(1,4,5)$.

Question 7 題目 7 Geometric interpretation of radicals | 根式的幾何詮釋

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Interpret the two square roots as distances from a point on the x-axis to two fixed points, and use reflection to convert the broken path into a straight segment.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Reflection 反射 | Distance Minimisation 距離最值

Solution 解答

Write

$$\sqrt{x^2+2x+5} = \sqrt{(x+1)^2+2^2}$$

and

$$\sqrt{x^2-6x+34} = \sqrt{(x-3)^2+5^2}.$$

Let $P = (x,0)$, $A = (-1,2)$, and $B = (3,5)$. The expression is $PA + PB$. Reflect A in the x-axis to $A' = (-1,-2)$. Since $PA = PA'$, the minimum of $PA + PB$ is the straight-line distance $A'B$:

$$A'B = \sqrt{(3+1)^2+(5+2)^2} = \sqrt{16+49} = \sqrt{65}.$$

Final Answer 最終答案. $\sqrt{65}$.

Question 8 題目 8 Angle bisector theorem | 角平分線定理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the 30°–60°–90° side ratio, followed by the angle bisector theorem.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Special Right Triangles 特殊直角三角形 | Angle Bisector Theorem 角平分線定理

Solution 解答

Since $\angle B = 30°$, the triangle is a 30°–60°–90° triangle. As $AB = \sqrt{3}$, we have

$$AC = 1, \qquad BC = 2.$$

By the angle bisector theorem,

$$\frac{AP}{PC} = \frac{AB}{BC} = \frac{\sqrt{3}}{2}.$$

Since $AP + PC = AC = 1$,

$$AP = \frac{\sqrt{3}}{\sqrt{3}+2} = 2\sqrt{3} - 3.$$

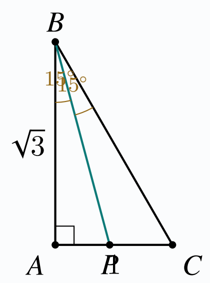

Figure 29. The angle bisector divides AC in the ratio $AB : BC = \sqrt{3} : 2$.

Final Answer 最終答案. $AP = 2\sqrt{3} - 3$.

Question 9 題目 9 Equal-area quadrilateral lemma 丨四邊形等面積引理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Place the origin at M and translate the equal-area conditions into equal cross products. Express two vertex vectors in a basis formed by the other two.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Vectors 向量** **Area and Cross Product 面積與叉積**

Solution 解答

Put M at the origin and denote the position vectors of A, B, C, D by

$$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}.$$

Because the quadrilateral is convex and M is inside it, the four oriented cross products have the same positive value:

$$\mathrm{a} \times \mathrm{b} = \mathrm{b} \times \mathrm{c} = \mathrm{c} \times \mathrm{d} = \mathrm{d} \times \mathrm{a} = K > 0.$$

Write

$$\mathrm{c} = \alpha\mathrm{a} + \beta\mathrm{b}.$$

Then

$$\mathrm{b} \times \mathrm{c} = \alpha(\mathrm{b} \times \mathrm{a}) = -\alpha K = K,$$

so $\alpha = -1$. Hence

$$\mathrm{c} = -\mathrm{a} + \beta\mathrm{b}.$$

Similarly, writing $\mathrm{d} = \gamma\mathrm{a} + \delta\mathrm{b}$ and using $\mathrm{d} \times \mathrm{a} = K$ gives $\delta = -1$, so

$$\mathrm{d} = \gamma\mathrm{a} - \mathrm{b}.$$

Now

$$K = \mathrm{c} \times \mathrm{d} = (-\mathrm{a} + \beta\mathrm{b}) \times (\gamma\mathrm{a} - \mathrm{b}) = K - \beta\gamma K.$$

Thus $\beta\gamma = 0$.
If $\beta = 0$, then $\mathrm{c} = -\mathrm{a}$, so M is the midpoint of AC and diagonal BD passes through it. If $\gamma = 0$, then $\mathrm{d} = -\mathrm{b}$, so M is the midpoint of BD and diagonal AC passes through it.

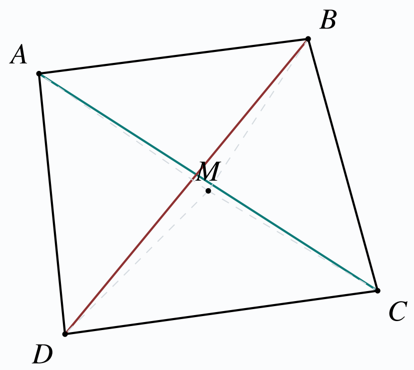

Figure 30. The four small triangles have equal area; the vector argument forces M to be the midpoint of one diagonal.

Final Answer 最終答案. Either M is the midpoint of AC or M is the midpoint of BD.

Question 10 題目 10 Prime arithmetic progressions | 質數等差數列

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Consider the six terms modulo 2, 3, and 5. If the common difference is not divisible by one of these primes, too many terms become divisible by that prime.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Modular Arithmetic 模算術** **Prime Progressions 質數等差數列**

Solution 解答

If d were odd, the six terms would alternate in parity, so at least three would be even. At most one prime can equal 2, hence this is impossible. Therefore $2 \mid d$.
If $3 \nmid d$, the residues modulo 3 repeat with period 3, so among six consecutive terms of the progression exactly two are divisible by 3. At most one of them can equal 3, a contradiction. Hence $3 \mid d$.
Now $6 \mid d$, so $d \geq 6$. If $5 \nmid d$, the first five terms represent all residue classes modulo 5. One of them is divisible by 5. If it is not the first term, then it is greater than 5, contradicting primality. If $p = 5$, then $p + 5d$ is also divisible by 5 and is greater than 5, again a contradiction. Thus $5 \mid d$.
Since $2, 3, 5$ are pairwise coprime,

$$30 \mid d.$$

Final Answer 最終答案. $30 \mid d$.

Question 11 題目 11 Double counting divisor sums | 約數和的雙重計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Expand every divisor sum and count the same divisor–multiple pairs by grouping first according to the multiple and then according to the divisor.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Double Counting 雙重計數 | Divisor Sums 約數和

Solution 解答

The left-hand side is

$$\sum_{m=1}^{n} \sum_{d \mid m} d.$$

Fix a positive integer $d \le n$. The integers $m \le n$ divisible by d are

$$d, 2d, \ldots, \left\lfloor \frac{n}{d} \right\rfloor d,$$

so there are $\lfloor n/d \rfloor$ of them. Therefore, regrouping the same terms by the divisor d,

$$\sum_{m=1}^{n} \sum_{d \mid m} d = \sum_{d=1}^{n} d \left\lfloor \frac{n}{d} \right\rfloor.$$

Renaming d as k gives the required identity.

Final Answer 最終答案. $\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k \left\lfloor \frac{n}{k} \right\rfloor$.

Question 12 題目 12 Discriminant controlled by a cubic | 由三次條件控制判別式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The number of real roots is determined by the discriminant. Compare the positive solution a with $\sqrt{8}$ using the cubic equation.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | **Discriminant 判別式** | **Root Location 根的位置**

Solution 解答

The discriminant is

$$\Delta = a^2 - 4(a^2 - 6) = 24 - 3a^2 = 3(8 - a^2).$$

Let

$$f(t) = t^3 - 6t - 6.$$

For $0 < t \le \sqrt{2}$, the function is decreasing and

$$f(t) \le f(0) = -6 < 0.$$

For $t > \sqrt{2}$, $f'(t) = 3t^2 - 6 > 0$, so f is strictly increasing. Moreover,

$$f(\sqrt{8}) = 2\sqrt{8} - 6 < 0, \qquad f(3) = 3 > 0.$$

Consequently the unique positive root a satisfies

$$a > \sqrt{8}.$$

Therefore $8 - a^2 < 0$, so $\Delta < 0$. The quadratic has no real roots.

Final Answer 最終答案. 0 real roots.

Question 13 題目 13 Median minimises absolute deviations | 中位數與絕對值和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The sum of absolute deviations is minimised at a median. Pair the terms symmetrically around the interval $[50, 51]$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | **Absolute Values 絕對值** | **Median Principle 中位數原理**

Solution 解答

Any $x \in [50, 51]$ is a median of the numbers $1, 2, \dots, 100$, so it minimises the sum. Pair k with $101 - k$ for $1 \le k \le 50$. For $x \in [50, 51]$,

$$|x - k| + |x - (101 - k)| = 101 - 2k.$$

Thus

$$\begin{aligned} \min f(x) &= \sum_{k=1}^{50} (101 - 2k) \\ &= 50 \cdot 101 - 2 \cdot \frac{50 \cdot 51}{2} \\ &= 2500. \end{aligned}$$

Final Answer 最終答案. 2500, attained for every $x \in [50, 51]$.

Question 14 題目 14 Cyclic divisibility classification | 循環整除分類

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Rotate the quadruple so that a is maximal. Then $b + c$ is either a or $2a$. The first case reduces to a short finite divisibility analysis, split according to whether $b \ge c$ or $b < c$.

Olympiad Knowledge Points 奧數知識點

- Number Theory 數論
- Divisibility 整除性
- Extremal Principle 極值原理
- Finite Classification 有限分類

Solution 解答

Because the conditions are invariant under cyclic rotation, rotate the quadruple so that

$$a = \max\{a, b, c, d\}.$$

Since $a \mid b + c$ and $b + c \le 2a$, either $b + c = a$ or $b + c = 2a$.
If $b + c = 2a$, then $b = c = a$. The remaining divisibility conditions force $d = a$, and the gcd condition gives

$$(a, b, c, d) = (1, 1, 1, 1).$$

Now suppose

$$a = b + c.$$

Case 1: $b \ge c$**.** Write $b = gB$, $c = gC$, where $\gcd(B, C) = 1$ and $B \ge C$. Since

$$c + d = mb$$

for some positive integer m, and $d \le a = b + c$, we have

$$(m-1)b \le 2c,$$

so $m \in \{1, 2, 3\}$. Moreover,

$$d = mb - c, \qquad c \mid d + a = (m+1)b,$$

so $C \mid m + 1$. Finally,

$$d \mid a + b = 2b + c$$

gives

$$mB - C \mid 2B + C.$$

The common factor g divides all four variables, so $g = 1$. Furthermore,

$$mB - C \mid m(2B + C) - 2(mB - C) = (m+2)C,$$

so only finitely many B need be checked. The admissible triples are

m	C	B	(a, b, c, d)
1	1	2	$(3, 2, 1, 1)$
1	1	4	$(5, 4, 1, 3)$
1	2	3	$(5, 3, 2, 1)$
1	2	5	$(7, 5, 2, 3)$
2	1	1	$(2, 1, 1, 1)$

Case 2: $b < c$. Since $c \mid d + a = d + b + c$, write

$$d + b = rc.$$

As $d \le b + c$ and $b < c$, we have $r \in \{1, 2\}$. Put $b = gB$, $c = gC$ with $\gcd(B, C) = 1$ and $B < C$. Then

$$d = rc - b, \qquad b \mid c + d = (r+1)c,$$

so $B \mid r + 1$. Also

$$rC - B \mid 2B + C.$$

Again $g = 1$. Also,

$$r(2B + C) - (rC - B) = (2r + 1)B,$$

so the check is finite. The admissible triples are

r	B	C	(a, b, c, d)
1	1	2	$(3, 1, 2, 1)$
1	1	4	$(5, 1, 4, 3)$
1	2	3	$(5, 2, 3, 1)$
1	2	5	$(7, 2, 5, 3)$
2	3	4	$(7, 3, 4, 5)$

Therefore all solutions are the cyclic rotations of the following eleven representatives:

$$(1,1,1,1), \quad (2,1,1,1), \quad (3,2,1,1), \quad (3,1,2,1), \quad (5,4,1,3), \quad (5,3,2,1),$$
$$(5,1,4,3), \quad (5,2,3,1), \quad (7,5,2,3), \quad (7,2,5,3), \quad (7,3,4,5).$$

Each listed quadruple is readily checked to satisfy all four divisibility conditions. The first has one distinct rotation; each of the other ten has four, so there are 41 ordered quadruples in total.

Final Answer 最終答案. All cyclic rotations of the eleven representatives above; equivalently, 41 ordered quadruples.

Question 15 題目 15 Symmetric rational identities | 對稱有理式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Normalise the variables by the sum $a+b+c$ and convert the condition into a relation between the elementary symmetric sums.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Normalisation 齊次化 | Symmetric Polynomials 對稱多項式

Solution 解答

Let

$$s = a+b+c, \qquad x = \frac{6a}{s}, \quad y = \frac{6b}{s}, \quad z = \frac{6c}{s}.$$

Then

$$x+y+z = 6,$$

and

$$a+b-5c = s-6c = s(1-z),$$

with analogous formulas for the other denominators. Hence

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z} = \frac{9}{5}.$$

Let

$$e_2 = xy+yz+zx, \qquad e_3 = xyz.$$

Since $x+y+z=6$,

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z} = \frac{e_2-9}{e_2-e_3-5}.$$

Thus

$$5(e_2-9) = 9(e_2-e_3-5),$$

which simplifies to

$$9e_3 = 4e_2.$$

Finally,

$$\begin{aligned}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) &= 6\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\ &= 6\frac{e_2}{e_3}\\ &= 6\cdot\frac{9}{4} = \frac{27}{2}.\end{aligned}$$

Final Answer 最終答案. $\dfrac{27}{2}$.

Question 16 題目 16 Lucas numbers and the golden ratio | 盧卡斯數與黃金比

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the conjugate of the golden ratio to identify the floor as a Lucas number, then apply the doubling identity for Lucas numbers.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Linear Recurrences 線性遞推 | Lucas Numbers 盧卡斯數 | Algebraic Conjugates 代數共軛

Solution 解答

Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2} = -\phi^{-1}.$$

The Lucas numbers are

$$L_m = \phi^m + \psi^m \in \mathbb{Z}.$$

For $m = 4n - 2$, m is even and

$$0 < \psi^m = \phi^{-m} < 1.$$

Therefore

$$\lfloor \phi^m \rfloor = L_m - 1,$$

so the required expression equals

$$L_m - 2.$$

Now put $k = 2n - 1$, so $m = 2k$ and k is odd. The Lucas doubling identity is

$$L_{2k} = L_k^2 - 2(-1)^k.$$

Since k is odd,

$$L_m = L_{2k} = L_k^2 + 2.$$

Thus

$$L_m - 2 = L_k^2 = L_{2n-1}^2,$$

which is a perfect square.

Final Answer 最終答案. The expression equals L_{2n-1}^2, hence is a perfect square.

2022/2023 | Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Exact trigonometric values | 特殊三角函數值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Write $15^\circ = 45^\circ - 30^\circ$ and apply the tangent subtraction formula.

Olympiad Knowledge Points 奧數知識點

Trigonometry 三角學 | **Angle Difference Formula 差角公式** | **Exact Values 精確值**

Solution 解答

Using the tangent subtraction formula,

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - 1/\sqrt{3}}{1 + 1/\sqrt{3}}.$$

Rationalising,

$$\tan 15^\circ = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}.$$

Final Answer 最終答案. $2 - \sqrt{3}$.

Question 2 題目 2 Change of logarithmic base | 對數換底

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
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Solution Strategy 思路解析

Take reciprocals of the logarithms by the change-of-base formula, then combine the logarithms into one product.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | **Logarithms 對數** | **Change of Base 換底公式**

Solution 解答

By change of base,

$$\frac{1}{\log_{xy}(xyz)} = \frac{\log(xy)}{\log(xyz)}.$$

Hence the required sum is

$$\frac{\log(xy) + \log(yz) + \log(zx)}{\log(xyz)}.$$

The numerator is

$$\log\big((xy)(yz)(zx)\big) = \log\big((xyz)^2\big) = 2\log(xyz).$$

Therefore the sum equals 2.

Final Answer 最終答案. 2.

Question 3 題目 3 Tangent addition identity | 正切和角恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Apply the tangent formula for the sum of three angles. The numerator must vanish because the total angle is π.

Olympiad Knowledge Points 奧數知識點

Trigonometry 三角學 | Tangent Addition 正切和角公式 | Symmetric Expressions 對稱式

Solution 解答

Let

$$A = \tan^{-1} x, \qquad B = \tan^{-1} y, \qquad C = \tan^{-1} z.$$

Since $A + B + C = \pi$, we have $\tan(A + B + C) = 0$. Therefore

$$x + y + z - xyz = 0,$$

so

$$x + y + z = xyz.$$

Hence

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = \frac{x + y + z}{xyz} = 1.$$

Final Answer 最終答案. 1.

Question 4 題目 4 Euler totient double counting 丨歐拉函數雙重計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Expand the floor as the number of multiples of k, then use the classical identity $\sum_{d|m}\varphi(d)=m$.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Euler Totient 歐拉函數 | Double Counting 雙重計數

Solution 解答

We have

$$\sum_{k=1}^{n}\varphi(k)\left\lfloor\frac{n}{k}\right\rfloor=\sum_{k=1}^{n}\varphi(k)\sum_{\substack{m\le n\\ k|m}}1.$$

Interchanging the order of summation gives

$$\sum_{m=1}^{n}\sum_{k|m}\varphi(k).$$

Using

$$\sum_{k|m}\varphi(k)=m,$$

we obtain

$$\sum_{m=1}^{n}m=\frac{n(n+1)}{2}.$$

Final Answer 最終答案. $\dfrac{n(n+1)}{2}$.

Question 5 題目 5 A product-constrained rational inequality 丨乘積限制有理不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Introduce three positive variables whose product is fixed. After rewriting the expression as a sum of reciprocal linear terms, clear denominators and use AM–GM.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 **Substitution 代換** **AM–GM 算術幾何平均**

Solution 解答

Set

$$x = \frac{2a^2}{bc}, \qquad y = \frac{2c^2}{ab}, \qquad z = \frac{2b^2}{ac}.$$

Then $x, y, z > 0$ and

$$xyz = 8.$$

The required expression becomes

$$S = \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}.$$

Now

$$S - 1 = \frac{x+y+z+2-xyz}{(x+1)(y+1)(z+1)}.$$

By AM–GM,

$$x + y + z \geq 3\sqrt[3]{xyz} = 6.$$

Since $xyz = 8$,

$$x + y + z + 2 - xyz \geq 6 + 2 - 8 = 0.$$

Thus $S \geq 1$. Equality holds when $x = y = z = 2$, which is equivalent to $a = b = c$.

Final Answer 最終答案. 1, attained when $a = b = c$.

Question 6 題目 6 Diagonal inequalities in a quadrilateral | 四邊形對角線不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let the diagonals meet at an interior point. Apply the triangle inequality to the four small triangles for the lower bound, and to two large triangles for the upper bound.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Triangle Inequality 三角不等式** **Convex Quadrilaterals 凸四邊形**

Solution 解答

Let the diagonals AC and BD meet at O. Since O lies inside the convex quadrilateral, the triangle inequalities in $\triangle AOB$ and $\triangle COD$ give

$$AB < AO + BO, \qquad CD < CO + DO.$$

Adding,

$$AB + CD < AC + BD.$$

Similarly, using $\triangle AOD$ and $\triangle BOC$,

$$AD + BC < AC + BD.$$

Therefore

$$\max(AB + CD, AD + BC) < AC + BD.$$

For the upper bound, apply the triangle inequality to the two possible boundary paths joining the endpoints of each diagonal:

$$2AC < (AB + BC) + (AD + DC),$$

and

$$2BD < (BA + AD) + (BC + CD).$$

Adding these two inequalities and dividing by 2 gives

$$AC + BD < AB + BC + CD + DA.$$

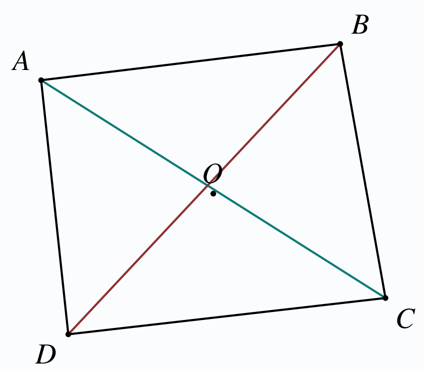

Figure 31. The diagonal intersection separates each side into a triangle-inequality comparison.

Final Answer 最終答案. $\max(AB + CD, AD + BC) < AC + BD < AB + BC + CD + DA.$

Question 7 題目 7 Sharp constant inequality | 最佳常數不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Divide by ab^2 and minimise the resulting expression. First minimise with respect to a using $a + 1/a \geq 2$, then minimise the remaining one-variable expression.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Best Constant 最佳常數 | AM–GM 算術幾何平均

Solution 解答

We seek the infimum of

$$R = \frac{(a+b)(ab+1)(b+1)}{ab^2}.$$

For fixed b,

$$\frac{(a+b)(ab+1)}{a} = ab + b^2 + 1 + \frac{b}{a} = b\left(a + \frac{1}{a}\right) + b^2 + 1.$$

Since $a + 1/a \geq 2$,

$$R \geq \frac{b+1}{b^2}(b^2 + 2b + 1) = \frac{(b+1)^3}{b^2}.$$

Now by AM–GM,

$$\frac{b+1}{3} = \frac{b/2 + b/2 + 1}{3} \geq \sqrt[3]{\frac{b^2}{4}},$$

so

$$\frac{(b+1)^3}{b^2} \geq \frac{27}{4}.$$

Equality holds when

$$a = 1, \qquad b = 2.$$

Therefore the largest possible constant is $27/4$.

Final Answer 最終答案. $k = \dfrac{27}{4}$.

Question 8 題目 8 Cubic norm identity | 三次範數恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Recognise the expression as the determinant of a circulant matrix. The product of two circulant matrices is circulant, and determinants multiply.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Polynomial Identities 多項式恆等式 | Circulant Matrices 循環矩陣 | Norm Multiplicativity 範數乘法性

Solution 解答

Define

$$F(a,b,c) = a^3 + b^3 + c^3 - 3abc.$$

We have the determinant representation

$$F(a,b,c) = \det\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Suppose

$$x = F(a,b,c), \qquad y = F(p,q,r),$$

where all six variables are integers. The product of the two circulant matrices is

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix} = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix},$$

where

$$A = ap + br + cq, \quad B = aq + bp + cr, \quad C = ar + bq + cp.$$

Thus A, B, C are integers, and

$$xy = \det(M_1)\det(M_2) = \det(M_1M_2) = F(A,B,C) \in \mathcal{S}.$$

Final Answer 最終答案. $\mathcal{S}$ is closed under multiplication.

Question 9 題目 9 Additive decomposition of an array ｜矩陣的可加分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Compare two permutation sums that differ only by swapping two selected columns. This yields the vanishing of every 2×2 alternating sum, which is exactly the condition for an additive row–column decomposition.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 | Matrices 矩陣 | Permutation Sums 排列和 | Functional Decomposition 函數分解

Solution 解答

For distinct rows i, k and distinct columns j, ℓ, choose a permutation that assigns j to row i and ℓ to row k. Compare it with the permutation obtained by swapping these two assignments while leaving all other rows unchanged. Since all permutation sums are equal,

$$a_{ij} + a_{k\ell} = a_{i\ell} + a_{kj}.$$

Thus every 2×2 alternating sum vanishes.
Set

$$x_i = a_{i1} - a_{11}, \qquad y_j = a_{1j}.$$

Taking $k = 1$ and $\ell = 1$ in the rectangle identity gives

$$a_{ij} + a_{11} = a_{i1} + a_{1j}.$$

Therefore

$$a_{ij} = a_{i1} - a_{11} + a_{1j} = x_i + y_j.$$

This proves the required representation.

Final Answer 最終答案. One valid choice is $x_i = a_{i1} - a_{11}$ and $y_j = a_{1j}$.

Question 10 題目 10 A central Delannoy identity | 中央 Delannoy 恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Apply Vandermonde's identity to the second binomial coefficient, interchange the order of summation, and use the absorption identity for binomial coefficients.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Binomial Coefficients 二項式係數** **Vandermonde Identity 范德蒙恆等式**

Solution 解答

By Vandermonde's identity,

$$\binom{n+k}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{k}{k-j} = \sum_{j=0}^{k} \binom{n}{j}\binom{k}{j}.$$

Therefore

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{n}{j}\binom{k}{j} \\ &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=j}^{n} \binom{n}{k}\binom{k}{j}.\end{aligned}$$

Using

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j},$$

we get

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} &= \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{k=j}^{n} \binom{n-j}{k-j} \\ &= \sum_{j=0}^{n} \binom{n}{j}^2 2^{n-j}.\end{aligned}$$

Replacing j by $n-k$ and using $\binom{n}{n-k} = \binom{n}{k}$ gives

$$\sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

Final Answer 最終答案. The two sums are equal for every positive integer n.

Question 11 題目 11 Classifying a recursive sequence | 遞推數列分類

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Subtract the index from the sequence. Monotonicity makes the new sequence nondecreasing, while the recurrence makes it constant on every interval $[n, 2n]$. Then eliminate all positive shifts using an elementary construction of a prime whose shifted index is composite.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Sequences 數列** **Prime Gaps 質數間隔** **Factorial Construction 階乘構造**

Solution 解答

Define

$$b_n = a_n - n.$$

Since $a_{n+1} \geq a_n + 1$, we have

$$b_{n+1} = a_{n+1} - (n+1) \geq a_n - n = b_n.$$

Thus (b_n) is nondecreasing. The recurrence gives

$$b_{2n} = a_{2n} - 2n = a_n + n - 2n = a_n - n = b_n.$$

Therefore, for every $n \leq k \leq 2n$,

$$b_n \leq b_k \leq b_{2n} = b_n,$$

so $b_k = b_n$. Since the intervals $[n, 2n]$ overlap and cover all positive integers, (b_n) is constant. Hence

$$a_n = n + c$$

for some integer $c \geq 0$.
The sequence $a_n = n$ ($c = 0$) clearly satisfies all conditions. We show that $c > 0$ is impossible. Let

$$N = (c+1)!.$$

Then the c consecutive integers

$$N+2, N+3, \ldots, N+c+1$$

are all composite. Let p be the first prime greater than $N + c + 1$. Then every integer from $N + 2$ to $p - 1$ is composite. Since

$$p - c \geq N + 2 \qquad \text{and} \qquad p - c < p,$$

the integer $p - c$ is composite. But

$$a_{p-c} = (p - c) + c = p$$

is prime, contradicting condition (iii).
Therefore $c = 0$ is the only possibility.

Final Answer 最終答案. $a_n = n$ for all $n \geq 1$.

Question 12 題目 12 Orthocenter vector identity | 垂心向量恆等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Take the circumcenter as the vector origin. The vector sum of the three vertex vectors is the orthocenter vector.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Vectors 向量** **Orthocenter 垂心** **Circumcenter 外心**

Solution 解答

Take O as the origin and write

$$\mathrm{a} = \overrightarrow{OA}, \qquad \mathrm{b} = \overrightarrow{OB}, \qquad \mathrm{c} = \overrightarrow{OC}.$$

Since A, B, C lie on the same circle centred at O,

$$|\mathrm{a}| = |\mathrm{b}| = |\mathrm{c}|.$$

Let

$$\mathrm{h} = \mathrm{a} + \mathrm{b} + \mathrm{c}.$$

Then

$$(\mathrm{h} - \mathrm{a}) \cdot (\mathrm{b} - \mathrm{c}) = (\mathrm{b} + \mathrm{c}) \cdot (\mathrm{b} - \mathrm{c}) = |\mathrm{b}|^2 - |\mathrm{c}|^2 = 0.$$

Thus the line through A and the point with vector h is perpendicular to BC. Similarly, it lies on the other two altitudes. Hence

$$\overrightarrow{OH} = \mathrm{h} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

Therefore

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OH} = 0.$$

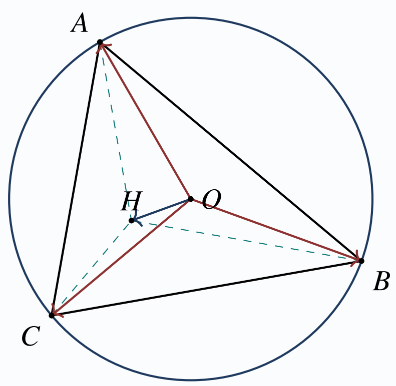

Figure 32. With O as origin, the orthocenter vector is $\mathrm{a} + \mathrm{b} + \mathrm{c}$.

Final Answer 最終答案. The zero vector 0.

2023/2024 ｜ Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Divisor counting | 約數個數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Factorise 2024 into prime powers, then apply the standard divisor-counting formula.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Prime Factorisation 質因數分解 | Divisor Function 約數函數

Solution 解答

We have

$$2024 = 8 \cdot 253 = 2^3 \cdot 11 \cdot 23.$$

Therefore the number of positive divisors is

$$(3+1)(1+1)(1+1) = 16.$$

Final Answer 最終答案. 16, so the correct choice is **D**.

Question 2 題目 2 A chain of inequalities | 連鎖不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use the last inequality to obtain a lower bound for x, and the middle inequality to obtain an upper bound.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Inequalities 不等式 | Domain Analysis 定義域分析

Solution 解答

Since $1/x$ appears and $x \geq 0$, we must have $x > 0$. From

$$\frac{1}{x} \leq 1$$

we get $x \geq 1$. On the other hand,

$$x \leq \frac{1}{x}$$

gives $x^2 \leq 1$, hence $x \leq 1$. Thus

$$x = 1$$

is the only solution.

Final Answer 最終答案. There is exactly 1 real number, so the correct choice is **B**.

Question 3 題目 3 Right angles in a convex octagon ｜凸八邊形的直角

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use exterior angles. A right interior angle contributes an exterior angle of $90°$, while every exterior angle of a strictly convex polygon is positive and their total is $360°$.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Convex Polygons 凸多邊形 | Exterior Angles 外角和

Solution 解答

If four interior angles were right angles, their corresponding exterior angles would already sum to

$$4 \cdot 90° = 360°.$$

The remaining four exterior angles would then have to be zero, contradicting strict convexity. Hence there can be at most three right angles.
The bound is attainable: choose three exterior angles equal to $90°$ and the remaining five equal to $18°$. These are all positive and sum to $360°$, so they can be realised by a convex octagon.

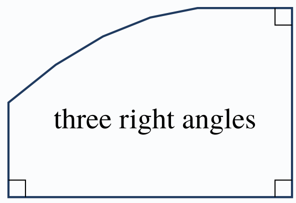

Figure 33. A convex octagon with exactly three right angles. 一個具有三個直角的凸八邊形示意圖。

Final Answer 最終答案. 3, so the correct choice is **C**.

Question 4 題目 4 Four-digit numbers with digit sum eight | 數字和為八的四位數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Subtract 1 from the first digit so that all four new variables are nonnegative, then use stars and bars.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Stars and Bars 隔板法** **Digit Problems 數位問題**

Solution 解答

Write the digits as d_1, d_2, d_3, d_4, where $d_1 \geq 1$ and

$$d_1 + d_2 + d_3 + d_4 = 8.$$

Set $e_1 = d_1 - 1 \geq 0$. Then

$$e_1 + d_2 + d_3 + d_4 = 7.$$

Since the total is only 7, no digit can exceed 9, so there are no further upper-bound restrictions. By stars and bars, the number of solutions is

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120.$$

Final Answer 最終答案. 120, so the correct choice is **C**.

Question 5 題目 5 A sum of squares equal to zero 丨平方和為零

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

A sum of two real squares is zero only when both squares are zero.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Non-negativity 非負性 | Linear Systems 線性方程組

Solution 解答

Both terms are nonnegative, so

$$2x - 3y = 0, \qquad 3x - 1 = 0.$$

Hence

$$x = \frac{1}{3}, \qquad y = \frac{2}{9},$$

and therefore

$$x + y = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}.$$

Final Answer 最終答案. $\frac{5}{9}$.

Question 6 題目 6 Equal selling prices with opposite percentage changes 丨同售價的盈虧問題

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Recover each cost price from the common selling price. Equal percentage gain and loss do not cancel when the cost prices are different.

Olympiad Knowledge Points 奧數知識點

Arithmetic 算術 | Percentages 百分率 | Profit and Loss 盈虧

Solution 解答

The cost price of the television sold at a 15% gain is

$$\frac{3910}{1.15} = 3400.$$

The cost price of the television sold at a 15% loss is

$$\frac{3910}{0.85} = 4600.$$

Thus the total cost and total revenue are

$$3400 + 4600 = 8000, \qquad 3910 + 3910 = 7820.$$

The dealer therefore makes a net loss of

$$8000 - 7820 = 180.$$

Final Answer 最終答案. A net loss of \$180.

Question 7 題目 7 One root is the square of the other | 一根為另一根的平方

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let the two roots be r and r^2. Use Vieta's formula to eliminate k, obtaining a cubic in r. The required sum can then be found from the elementary symmetric sums of the cubic roots, without solving the cubic explicitly.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Vieta's Formula 韋達定理 | Symmetric Sums 根的對稱和

Solution 解答

Let the roots be r and r^2. Vieta's formula gives

$$r + r^2 = k + 1, \qquad r^3 = 3k + 1.$$

Eliminating k yields

$$r^3 - 3r^2 - 3r + 2 = 0.$$

This cubic has one root in each of the intervals $(-2, -1)$, $(0, 1)$ and $(3, 4)$, so it has three distinct real roots r_1, r_2, r_3. The corresponding values

$$k_i = r_i^2 + r_i - 1$$

are also distinct: if $k_i = k_j$, then $(r_i - r_j)(r_i + r_j + 1) = 0$, so $r_i + r_j = -1$; the third root would then be 4, but 4 is not a root of the cubic.
For the cubic

$$r^3 - 3r^2 - 3r + 2 = 0,$$

Vieta's formula gives

$$\sum r_i = 3, \qquad \sum_{i<j} r_i r_j = -3.$$

Hence

$$\sum r_i^2 = \left(\sum r_i\right)^2 - 2\sum_{i<j} r_i r_j = 9 + 6 = 15.$$

Therefore

$$\sum k_i = \sum (r_i^2 + r_i - 1) = 15 + 3 - 3 = 15.$$

Final Answer 最終答案. 15.

Question 8 題目 8 Coloured balls and a linear system | 彩球與線性方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Translate the two verbal conditions into linear equations and parametrise all positive integer solutions.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Linear Diophantine Equations 線性丟番圖方程** **Parity 奇偶性**

Solution 解答

Let the numbers of red, white and blue balls be r, w, b. Then

$$w + b = 4r, \qquad r + b = 6w.$$

From the first equation $b = 4r - w$. Substituting into the second gives

$$5r = 7w.$$

Thus, for some positive integer t,

$$r = 7t, \qquad w = 5t, \qquad b = 23t.$$

The total is

$$r + w + b = 35t.$$

It is even, so t is even; it is less than 100, so $35t < 100$. Therefore $t = 2$, giving

$$35t = 70.$$

Final Answer 最終答案. 70 balls.

Question 9 題目 9 Area of a right trapezium | 直角梯形面積

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First use the area of the right triangle ABC to find AB. Then place the trapezium in coordinates and use $CD = DA$ to determine the upper base.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Coordinate Geometry 坐標幾何 | Area 面積

Solution 解答

Since $\triangle ABC$ is right-angled at B,

$$384 = \frac{1}{2} \cdot AB \cdot 24,$$

so

$$AB = 32.$$

Place

$$A = (0,0), \quad B = (32,0), \quad C = (32,24), \quad D = (d,24).$$

Then

$$CD = 32 - d, \qquad DA = \sqrt{d^2 + 24^2}.$$

The condition $CD = DA$ gives

$$(32-d)^2 = d^2 + 576,$$

so

$$1024 - 64d = 576, \qquad d = 7.$$

Hence $CD = 25$, and the area of the trapezium is

$$\frac{AB + CD}{2} \cdot BC = \frac{32+25}{2} \cdot 24 = 684.$$

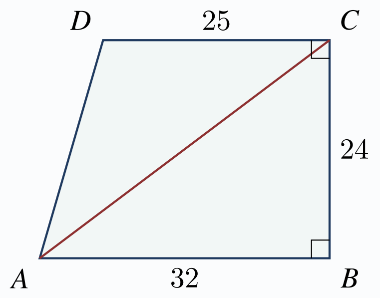

Figure 34. The coordinate model of the trapezium. 梯形的坐標示意圖。

Final Answer 最終答案. 684.

Question 10 題目 10 Squares on an equilateral triangle 丨等邊三角形三邊上的正方形

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Choose coordinates for the equilateral triangle. The outer vertices X, Y, Z are then obtained by rotating the side vectors through 90°. Direct calculation shows that $\triangle XYZ$ is equilateral.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 **Coordinate Geometry 坐標幾何** **Rotations 旋轉變換**

Solution 解答

Take

$$A = (0,0), \qquad B = (1,0), \qquad C = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

The relevant outer vertices of the three squares are

$$X = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad Y = \left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right), \quad Z = (1, -1).$$

For example,

$$\begin{aligned} XY^2 &= \left(\frac{1+2\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 4 + \sqrt{3}. \end{aligned}$$

The same calculation, or the 120° rotational symmetry of the whole configuration, gives

$$XY = YZ = ZX = \sqrt{4+\sqrt{3}}.$$

Thus $\triangle XYZ$ is equilateral and its perimeter is

$$3\sqrt{4+\sqrt{3}}.$$

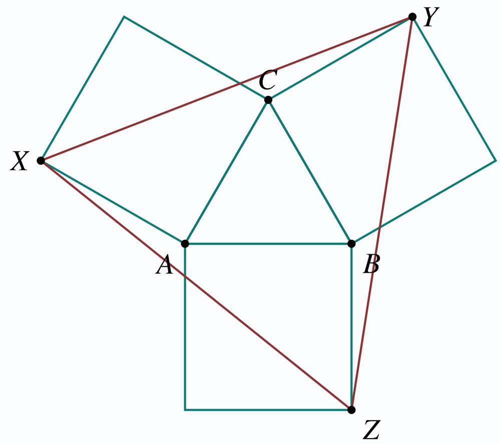

Figure 35. $\triangle XYZ$ is equilateral by the 120° rotational symmetry. 由 120° 旋轉對稱可見 $\triangle XYZ$ 為等邊三角形。

Final Answer 最終答案. $3\sqrt{4+\sqrt{3}}$.

Question 11 題目 11 A cyclic cubic system | 循環三次方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use $c = -a - b$ to reduce the second equation to a polynomial in a and b. The resulting expression is a negative perfect square.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Factorisation 因式分解** **Cyclic Expressions 循環式**

Solution 解答

From $a+b+c=0$, we have $c=-a-b$. Substituting into the second equation and simplifying,

$$\begin{aligned} ab^3+bc^3+ca^3 &= ab^3+b(-a-b)^3+(-a-b)a^3 \\ &= -\left(a^2+ab+b^2\right)^2. \end{aligned}$$

Therefore

$$a^2+ab+b^2=0.$$

But

$$a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3}{4}b^2\geq 0,$$

with equality only when $a=b=0$. Hence $c=0$ as well.

Final Answer 最終答案. $(a,b,c)=(0,0,0)$.

Question 12 題目 12 Mutual divisibility conditions | 互相整除條件

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Write $a-1=kb$. The second divisibility condition then gives a strong size bound on k, leaving only $k=1$ and $k=2$.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Divisibility 整除性** **Diophantine Equations 丟番圖方程**

Solution 解答

Write

$$a-1=kb, \qquad k\geq 1.$$

Then

$$a=kb+1, \qquad 2a+1=2kb+3.$$

Since $2a+1$ divides the positive integer $5b-3$, we must have

$$2kb+3\leq 5b-3.$$

This is impossible for $k\geq 3$, so $k=1$ or $k=2$.
If $k=1$, then $a=b+1$ and

$$2b+3 \mid 5b-3.$$

Since $(5b-3)/(2b+3)<5/2$, the quotient is 1 or 2:

$$5b-3=2b+3 \quad\Rightarrow\quad b=2,$$

or

$$5b - 3 = 2(2b + 3) \quad \Rightarrow \quad b = 9.$$

These give $(a, b) = (3, 2)$ and $(10, 9)$.
If $k = 2$, then $a = 2b + 1$ and

$$4b + 3 \mid 5b - 3.$$

The quotient is less than $5/4$, so it must equal 1. Hence

$$5b - 3 = 4b + 3, \qquad b = 6,$$

giving $(a, b) = (13, 6)$.

Final Answer 最終答案. $(a, b) = (3, 2), (10, 9), (13, 6)$.

Question 13 題目 13 Angle bisector and perpendicular bisector | 角平分線與垂直平分線

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Editorial Note 編按

The ratio is not a universal numerical constant; it depends on the side lengths of $\triangle ABC$. The data determine the following exact expression. In the diagram E lies beyond B, corresponding to $AC > BC$.

Solution Strategy 思路解析

Place the angle bisector CD on the x-axis. The coordinates of D follow from the angle-bisector theorem, and the perpendicular bisector of CD becomes a vertical line.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Angle Bisector Theorem 角平分線定理 | Coordinate Geometry 坐標幾何

Solution 解答

Let

$$AC = p, \qquad BC = q, \qquad p > q,$$

and write $\angle ACD = \angle DCB = \theta$. Choose coordinates

$$C = (0, 0), \quad A = (p\cos\theta, p\sin\theta), \quad B = (q\cos\theta, -q\sin\theta).$$

By the angle-bisector theorem,

$$\frac{AD}{DB} = \frac{p}{q},$$

so

$$D = \left(\frac{2pq\cos\theta}{p+q}, 0\right).$$

The perpendicular bisector of CD is therefore

$$x = \frac{pq\cos\theta}{p+q}.$$

Intersecting this line with AB gives

$$E = \left(\frac{pq\cos\theta}{p+q}, -\frac{pq\sin\theta}{p-q}\right).$$

A direct calculation yields

$$DA^2 = \frac{p^2}{(p+q)^2}\left((p-q)^2\cos^2\theta + (p+q)^2\sin^2\theta\right),$$

while

$$ED^2 = \frac{p^2q^2}{(p-q)^2(p+q)^2}\left((p-q)^2\cos^2\theta + (p+q)^2\sin^2\theta\right).$$

Consequently,

$$\frac{ED}{DA} = \frac{q}{p-q} = \frac{BC}{AC-BC}.$$

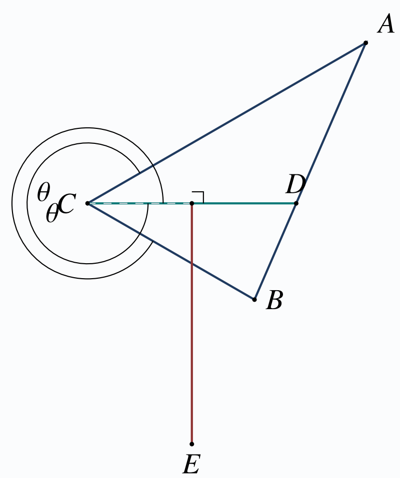

Figure 36. Coordinate placement with CD horizontal and its perpendicular bisector vertical.
將 CD 放在水平軸上的坐標配置。

Final Answer 最終答案. $\dfrac{ED}{DA} = \dfrac{BC}{AC-BC}$ for the configuration shown.

Question 14 題目 14 A locus defined by a 45° angle ｜四十五度圓周角軌跡

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The locus of points from which the fixed segment AB subtends a $45°$ angle is an arc of a circle. Find the two possible circles and then minimise the distance from C to the relevant circle.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Circle Locus 圓軌跡 | Triangle Inequality 三角不等式

Solution 解答

Place

$$B = (0,0), \qquad A = (0,2), \qquad C = (3,0).$$

A circle through A and B on which chord AB subtends an inscribed angle of $45°$ has central angle $90°$. Its radius is therefore

$$R = \frac{AB}{2\sin 45°} = \sqrt{2}.$$

The two possible centres are

$$O_1 = (1,1), \qquad O_2 = (-1,1).$$

The circle centred at O_1 is closer to C, since

$$CO_1 = \sqrt{5}, \qquad CO_2 = \sqrt{17}.$$

For every point D on the relevant arc of the first circle,

$$CD \geq CO_1 - O_1D = \sqrt{5} - \sqrt{2}.$$

Equality holds at the point of the circle lying on the ray from O_1 to C. This point lies on the major arc AB, so indeed $\angle ADB = 45°$.

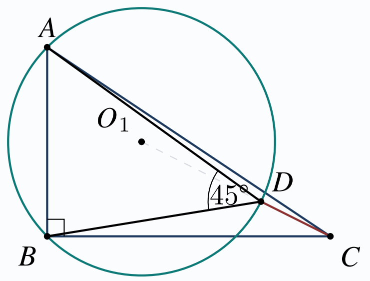

Figure 37. The equality point is the point of the locus circle nearest to C. 等號點為軌跡圓上最接近 C 的點。

Final Answer 最終答案. $\sqrt{5} - \sqrt{2}$.

Question 15 題目 15 A two-variable rational inequality ｜二元分式不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Apply the Engel form of Cauchy–Schwarz to combine the two fractions into a single expression depending only on $s = x + y$. Since $x, y > 1$, we have $s > 2$; subtract 8 from the resulting rational function and factor the numerator as $(s - 4)^2$.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Cauchy–Schwarz 柯西不等式 | Equality Cases 等號條件

Solution 解答

By Cauchy's inequality,

$$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq \frac{(x+y)^2}{x+y-2}.$$

Let $s = x + y > 2$. Then

$$\frac{s^2}{s-2} - 8 = \frac{s^2 - 8s + 16}{s-2} = \frac{(s-4)^2}{s-2} \geq 0.$$

Therefore the required inequality holds. Equality in both steps occurs when

$$x = y = 2.$$

Final Answer 最終答案. The minimum value is 8, attained at $x = y = 2$.

Question 16 題目 16 Maximising a sum of reciprocals ｜單位分數和的最大值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The smallest denominator c controls the size of the sum. Split into the cases $c = 2$, $c = 3$, and $c \geq 4$, and in each case choose the remaining denominators as small as the strict inequality permits.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Unit Fractions 單位分數 | Extremal Principle 極值原理

Solution 解答

Clearly $c \geq 2$.
If $c \geq 4$, then

$$S \leq \frac{3}{c} \leq \frac{3}{4} < \frac{41}{42}.$$

If $c = 3$ and $b = 3$, the strict inequality gives $a > 3$, so

$$S \leq \frac{1}{4} + \frac{1}{3} + \frac{1}{3} = \frac{11}{12} < \frac{41}{42}.$$

If $c = 3$ and $b \geq 4$, then

$$S \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{5}{6}.$$

Now let $c = 2$. We cannot have $b = 2$. If $b = 3$, then

$$\frac{1}{a} + \frac{1}{3} + \frac{1}{2} < 1$$

forces $a \geq 7$, and the largest value in this case is

$$\frac{1}{7} + \frac{1}{3} + \frac{1}{2} = \frac{41}{42}.$$

If $b = 4$, then $a \geq 5$ and

$$S \leq \frac{1}{5} + \frac{1}{4} + \frac{1}{2} = \frac{19}{20} < \frac{41}{42}.$$

If $b \geq 5$, then

$$S \leq \frac{1}{2} + \frac{2}{5} = \frac{9}{10}.$$

Thus the overall maximum is attained at $(a, b, c) = (7, 3, 2)$.

Final Answer 最終答案. $\max S(a, b, c) = \frac{41}{42}$, attained at $(7, 3, 2)$.

2023/2024 ｜ Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Iteration of a rational radical function 丨根式函數的迭代

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Compute the first few iterates and observe that the coefficient of x^2 in the denominator increases by one at each step. Then prove the pattern by induction.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Functional Iteration 函數迭代 | Mathematical Induction 數學歸納法

Solution 解答

We claim that

$$f_n(x) = \frac{x}{\sqrt{1+nx^2}}.$$

The formula is true for $n = 1$. If it holds for n, then

$$\begin{aligned} f_{n+1}(x) &= f\left(\frac{x}{\sqrt{1+nx^2}}\right) \\ &= \frac{x/\sqrt{1+nx^2}}{\sqrt{1+\dfrac{x^2}{1+nx^2}}} \\ &= \frac{x}{\sqrt{1+(n+1)x^2}}. \end{aligned}$$

Thus the formula follows for every positive integer n.

Final Answer 最終答案. $f_n(x) = \dfrac{x}{\sqrt{1+nx^2}}$, so the correct choice is **D**.

Question 2 題目 2 Logarithms with different bases 丨不同底數的對數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let the three expressions have a common value t. Rewrite 3^t, 5^t and 15^t in terms of a and b; the key is that $15^t = 3^t 5^t$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 **Logarithms 對數** **Change of Representation 表示轉換**

Solution 解答

Let the common value be t. Then

$$3^t = 9a, \qquad 5^t = 125b, \qquad 15^t = a + b.$$

Since $15^t = 3^t 5^t$, we obtain

$$a + b = (9a)(125b) = 1125ab.$$

Dividing by ab gives

$$\frac{1}{a} + \frac{1}{b} = 1125.$$

Final Answer 最終答案. 1125, so the correct choice is **B.**

Question 3 題目 3 A product recurrence and a weighted geometric sum 丨乘積遞推與加權等比和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

First obtain a closed form for a_n by telescoping the product. Then evaluate the weighted geometric sum $\sum(k+1)2^{k-1}$.

Olympiad Knowledge Points 奧數知識點

Sequences 數列 **Telescoping Products 連乘消去** **Weighted Geometric Sums 加權等比和**

Solution 解答

For $n \geq 1$,

$$\begin{aligned} a_n &= 2\prod_{k=1}^{n-1} \frac{2(k+2)}{k+1} \\ &= 2^n \cdot \frac{n+1}{2} = (n+1)2^{n-1}. \end{aligned}$$

Also,

$$\sum_{k=1}^{m}(k+1)2^{k-1} = m2^m,$$

which follows by adding the standard identities for $\sum k2^{k-1}$ and $\sum 2^{k-1}$, or by induction. Therefore

$$a_1 + \cdots + a_{2023} = 2023 \cdot 2^{2023},$$

while

$$a_{2024} = 2025 \cdot 2^{2023}.$$

Hence the required ratio is

$$\frac{2025}{2023}.$$

Final Answer 最終答案. $\frac{2025}{2023}$, so the correct choice is **C**.

Question 4 題目 4 Repeated differences and the pigeonhole principle | 重複差值與鴿巢原理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Assume every difference occurs at most 14 times. Minimise the sum of 99 positive differences under that restriction and compare it with $a_{100} - a_1 \le 399$.

Olympiad Knowledge Points 奧數知識點

Combinatorics 組合 **Pigeonhole Principle 鴿巢原理** **Extremal Principle 極值原理**

Solution 解答

Suppose, to the contrary, that every positive integer occurs among the d_i at most 14 times. To make the sum of the 99 differences as small as possible, we would take fourteen copies of each of

$$1, 2, \ldots, 7$$

and one additional copy of 8. Therefore

$$\sum_{i=1}^{99} d_i \ge 14(1+2+\cdots+7) + 8 = 14 \cdot 28 + 8 = 400.$$

But

$$\sum_{i=1}^{99} d_i = a_{100} - a_1 \le 400 - 1 = 399,$$

a contradiction. Hence some value occurs at least 15 times.

Final Answer 最終答案. There exists an integer n such that $d_i = n$ for at least 15 indices.

Question 5 題目 5 Maximum of a sum of square roots | 根式和的最大值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

The largest admissible k is the maximum value of the left-hand side on its domain. Apply Cauchy's inequality to the two square roots.

Olympiad Knowledge Points 奧數知識點

Inequalities 不等式 | Cauchy–Schwarz 柯西不等式 | Optimisation 最值

Solution 解答

The domain is $3 \le x \le 6$. By Cauchy's inequality,

$$\left(\sqrt{x-3}+\sqrt{6-x}\right)^2 \le 2\big((x-3)+(6-x)\big) = 6.$$

Hence

$$\sqrt{x-3}+\sqrt{6-x} \le \sqrt{6}.$$

Equality holds when

$$x-3 = 6-x,$$

that is, when $x = 9/2$. Thus the maximum is $\sqrt{6}$.

Final Answer 最終答案. $k_{\max} = \sqrt{6}$.

Question 6 題目 6 A hidden perfect square | 隱藏的完全平方數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Clear denominators and group the cubic terms using

$$a^3+b^3-ab(a+b) = (a+b)(a-b)^2.$$

The equation then collapses to an explicit square.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Algebraic Factorisation 代數因式分解**
Perfect Squares 完全平方數

Solution 解答

Multiplying the given equation by ab gives

$$a(a^2 - a - c) + b(b^2 - b - c) = ab(a + b + 2).$$

After moving all terms to one side,

$$a^3 + b^3 - a^2b - ab^2 - a^2 - b^2 - 2ab - c(a + b) = 0.$$

Now

$$a^3 + b^3 - a^2b - ab^2 = (a + b)(a - b)^2$$

and

$$a^2 + b^2 + 2ab = (a + b)^2.$$

Therefore

$$(a + b)(a - b)^2 - (a + b)^2 - c(a + b) = 0.$$

Since $a + b > 0$, division by $a + b$ yields

$$a + b + c = (a - b)^2.$$

Thus $a + b + c$ is a perfect square.

Final Answer 最終答案. $a + b + c = (a - b)^2$.

Question 7 題目 7 Quadratic residues modulo an odd prime | 奇素數模下的二次剩餘

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Work modulo p and divide by the nonzero residues b and d. The two ratios a/b and c/d are both square roots of -1, so they are either equal or negatives of one another.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 **Modular Arithmetic 模算術** **Quadratic Residues 二次剩餘**

Solution 解答

All of a, b, c, d are nonzero modulo p. Put

$$r \equiv ab^{-1} \pmod{p}, \qquad s \equiv cd^{-1} \pmod{p}.$$

The assumptions give

$$r^2 \equiv s^2 \equiv -1 \pmod{p}.$$

Thus

$$(r-s)(r+s) \equiv 0 \pmod{p},$$

so $s \equiv r$ or $s \equiv -r \pmod{p}$.
If $s \equiv r$, then $a \equiv rb$ and $c \equiv rd$, so

$$ac + bd \equiv (r^2+1)bd \equiv 0 \pmod{p},$$

whereas

$$ad + bc \equiv 2rbd \not\equiv 0 \pmod{p},$$

because p is odd and r, b, d are nonzero.
If $s \equiv -r$, then

$$ad + bc \equiv rbd - rbd \equiv 0 \pmod{p},$$

while

$$ac + bd \equiv (-r^2+1)bd \equiv 2bd \not\equiv 0 \pmod{p}.$$

Hence exactly one of the two expressions is divisible by p.

Final Answer 最終答案. Exactly one of $ac + bd$ and $ad + bc$ is a multiple of p.

Question 8 題目 8 Divisibility of every consecutive block | 所有連續區段的整除性

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Compare adjacent blocks to obtain periodic congruences. Then combine the conditions for block lengths $m-1$ and $2m-1$ to force the first term to be divisible by every positive integer.

Olympiad Knowledge Points 奧數知識點

Number Theory 數論 | Congruences 同餘 | Consecutive Sums 連續和

Solution 解答

Assume such a sequence exists. Fix an integer $m \geq 2$.

Every sum of $m-1$ consecutive terms is divisible by m. Subtracting two adjacent such sums gives

$$a_{n+m-1} \equiv a_n \pmod{m} \qquad (n \geq 1). \tag{1}$$

Now consider the block of $2m-1$ terms

$$a_2 + a_3 + \cdots + a_{2m}.$$

It is divisible by $2m$, hence by m. It can be split as

$$(a_2 + \cdots + a_m) + (a_{m+1} + \cdots + a_{2m-1}) + a_{2m}.$$

Each parenthesised sum contains $m-1$ consecutive terms and is divisible by m. Consequently,

$$m \mid a_{2m}. \tag{2}$$

Next apply (1) with m replaced by $2m$ and $n = 1$. We obtain

$$a_{2m} \equiv a_1 \pmod{2m}.$$

Together with (2), this implies $m \mid a_1$. Since $m \geq 2$ was arbitrary, a_1 would be divisible by every positive integer, impossible for a positive integer.

Final Answer 最終答案. No such sequence exists.

Question 9 題目 9 A cyclic system of four positive variables | 四個正變量的循環方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Let the common ratio be λ. Subtract opposite equations to prove $x = z$ and $y = w$. The remaining equation factors into $(x-y)(xy-1) = 0$.

Olympiad Knowledge Points 奧數知識點

Algebra 代數 | Cyclic Systems 循環方程組 | Factorisation 因式分解

Solution 解答

Let the common value of the four fractions be $\lambda > 0$. Then

$$xyz + 1 = \lambda(x+1), \qquad zwx + 1 = \lambda(z+1),$$

so subtraction gives

$$xz(y-w) = \lambda(x-z). \tag{1}$$

Similarly, subtracting the second and fourth equations gives

$$yw(z-x) = \lambda(y-w). \tag{2}$$

Multiply (1) by λ and use (2):

$$\lambda xz(y-w) = \lambda^2(x-z),$$

while

$$\lambda(y-w) = -yw(x-z).$$

Hence

$$-xyzw(x-z) = \lambda^2(x-z),$$

and therefore

$$(\lambda^2 + xyzw)(x-z) = 0.$$

All quantities are positive, so $x = z$. Equation (1) then gives $y = w$.
The first two fractions are now equal:

$$\frac{x^2y+1}{x+1} = \frac{xy^2+1}{y+1}.$$

Cross multiplication and factorisation give

$$(x-y)(xy-1) = 0.$$

If $x = y$, then all four variables equal 12.
If $xy = 1$, then the sum condition gives

$$2(x+y) = 48, \qquad x+y = 24.$$

Thus x and y are the two roots of

$$t^2 - 24t + 1 = 0,$$

namely

$$12 \pm \sqrt{143}.$$

Both alternating orders are possible.

Final Answer 最終答案.

$$(x, y, z, w) = (12, 12, 12, 12),$$

or

$$(12+\sqrt{143}, 12-\sqrt{143}, 12+\sqrt{143}, 12-\sqrt{143}),$$

or the tuple obtained by interchanging the two alternating values.

Question 10 題目 10 Incenter, median and circumcircle ｜內心、中線與外接圓

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Use coordinates adapted to the angle bisector BI: place B at the origin and the internal bisector of $\angle B$ on the x-axis. This makes I and E lie on the same axis and permits direct computation of D.

Olympiad Knowledge Points 奧數知識點

Geometry 幾何 | Coordinate Geometry 坐標幾何 | Incenter 内心 | Circumcircle 外接圓

Solution 解答

Use the standard side notation

$$a = BC, \qquad b = CA, \qquad c = AB, \qquad s = \frac{a+b+c}{2}.$$

Write $\angle B = 2\theta$ and choose coordinates

$$B = (0,0), \quad A = (c\cos\theta, c\sin\theta), \quad C = (a\cos\theta, -a\sin\theta).$$

The x-axis is the internal angle bisector of $\angle B$. The incenter and midpoint are

$$I = \left(\frac{ac\cos\theta}{s}, 0\right), \qquad M = \left(\frac{a\cos\theta}{2}, -\frac{a\sin\theta}{2}\right).$$

The circumcircle through B has equation

$$x^2 + y^2 - \frac{a+c}{2\cos\theta}x + \frac{a-c}{2\sin\theta}y = 0.$$

Its second intersection with the x-axis is therefore

$$E = \left(\frac{a+c}{2\cos\theta}, 0\right).$$

Solving the two line equations for MI and AC gives

$$D = \left(\frac{a(2c-b)\cos\theta}{a-b+c}, \frac{ab\sin\theta}{a-b+c}\right).$$

By the cosine law,

$$4ac\cos^2\theta = (a+c)^2 - b^2 = (a-b+c)(a+b+c). \tag{1}$$

Using (1) to simplify the squared distances, we obtain

$$EI^2 = \frac{ab^2c}{(a-b+c)(a+b+c)},$$

$$ED^2 = \frac{ab^3(a+b-c)}{(a-b+c)^2(a+b+c)},$$

while

$$IB^2 = \frac{ac(a-b+c)}{a+b+c}, \qquad IC^2 = \frac{ab(a+b-c)}{a+b+c}.$$

Consequently,

$$\left(\frac{ED}{EI}\right)^2 = \frac{b(a+b-c)}{c(a-b+c)} = \left(\frac{IC}{IB}\right)^2.$$

All lengths are positive, so the required equality follows.

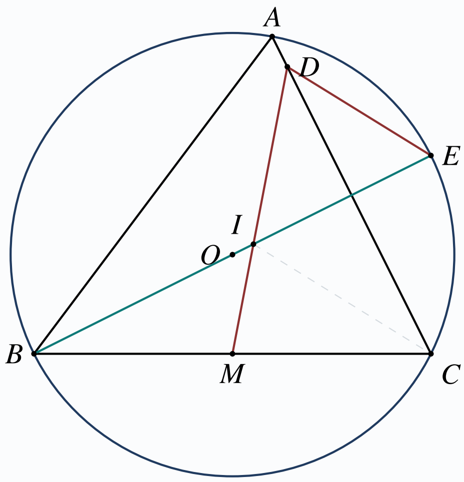

Figure 38. The configuration for Question 10, drawn from a representative acute triangle.
第 10 題配置的講義式示意圖。

Final Answer 最終答案. $\dfrac{ED}{EI} = \dfrac{IC}{IB}$.

Question 11 題目 11 A nonlinear recurrence inequality ｜非線性數列不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

Rewrite the condition in terms of $(a_n-1)(a_{n+2}-1)$. If a term exceeds 1, exactly one of its two neighbours must also exceed 1. Then show that two adjacent terms greater than 1 lead to contradictory inequalities.

Olympiad Knowledge Points 奧數知識點

Sequences 數列 **Inequalities 不等式** **Contradiction 反證法**

Solution 解答

The given inequality is equivalent to

$$(a_n - 1)(a_{n+2} - 1) \le 1 - a_{n+1}^2. \tag{1}$$

We first show that no two adjacent terms a_k, a_{k+1} with $k \geq 2$ can both exceed 1.
Suppose $a_k > 1$ and $a_{k+1} > 1$. Applying (1) with $n = k - 1$, the right-hand side is negative, so $a_{k-1} < 1$, and

$$(a_{k+1} - 1)(1 - a_{k-1}) \geq a_k^2 - 1.$$

Since $0 < 1 - a_{k-1} < 1$, it follows that

$$a_{k+1} - 1 > a_k^2 - 1 > a_k - 1,$$

so $a_{k+1} > a_k$.
Applying (1) with $n = k$, we similarly get $a_{k+2} < 1$ and

$$(a_k - 1)(1 - a_{k+2}) \geq a_{k+1}^2 - 1.$$

Thus

$$a_k - 1 > a_{k+1}^2 - 1 > a_{k+1} - 1,$$

so $a_k > a_{k+1}$, a contradiction.
Now suppose $a_n > 1$ for some $n \geq 3$. Applying (1) with centre term a_n, namely with index $n - 1$, gives

$$(a_{n-1} - 1)(a_{n+1} - 1) < 0.$$

Hence exactly one of a_{n-1}, a_{n+1} is greater than 1. This produces an adjacent pair greater than 1 whose lower index is at least 2, contradicting the result above.
Therefore $a_n \leq 1$ for every $n \geq 3$, and in particular

$$a_{2023} \leq 1.$$

Final Answer 最終答案. $a_{2023} \leq 1$.

2024/2025 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Telescoping radicals | 望遠鏡根式和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

將每項有理化。由於相鄰根號內的數相差 4，每項會化成兩個相鄰平方根之差的四分之一，形成望遠鏡相消。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 根式有理化。
- 望遠鏡級數。
- 首尾項與項數辨認。

Solution 解答

For $k = 0, 1, \dots, 505$,

$$\frac{1}{\sqrt{4k+1}+\sqrt{4k+5}} = \frac{\sqrt{4k+5}-\sqrt{4k+1}}{4}.$$

Hence

$$\begin{aligned} S &= \frac{1}{4}\left[(\sqrt{5}-\sqrt{1}) + (\sqrt{9}-\sqrt{5}) + \cdots + (\sqrt{2025}-\sqrt{2021})\right] \\ &= \frac{\sqrt{2025}-\sqrt{1}}{4} = \frac{45-1}{4} = 11. \end{aligned}$$

Final Answer 最終答案. 11, so the correct choice is **D**.

Question 2 題目 2 Double counting | 重複計數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把三種才藝的「人次」相加。同一位學生若有一項才藝就被計一次，有兩項才藝就被計兩次。利用「總人數」與「總才藝人次」的差，即可得到恰有兩項才藝的人數。

Olympiad Knowledge Points 奧數知識點

組合 Combinatorics

- 補集計數。
- 雙重計數與集合重疊。
- 恰有一項／兩項性質的分類。

Solution 解答

The numbers who can sing, dance, and act are respectively

$$100 - 42 = 58, \qquad 100 - 65 = 35, \qquad 100 - 29 = 71.$$

Thus the total number of talent memberships is

$$58 + 35 + 71 = 164.$$

Let x students have exactly two talents and y students have exactly one talent. Since every student has at least one talent and nobody has all three,

$$x + y = 100, \qquad 2x + y = 164.$$

Subtracting gives $x = 64$.

Final Answer 最終答案. 64 students; therefore the correct choice is **E** (none of the listed numbers).

Question 3 題目 3 Symmetric expressions ｜對稱式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

常數項為 1，適合把方程除以 x，直接取得 $x + 1/x$。再利用立方恆等式求 $x^3 + 1/x^3$，毋須解出根式形式的 x。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 倒數對稱式。
- $u^3 + v^3 = (u + v)^3 - 3uv(u + v)$。
- 避免不必要的求根。

Solution 解答

Since $x \neq 0$, divide the equation by x:

$$x + \frac{1}{x} = -4.$$

Therefore

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) \\ &= (-4)^3 - 3(-4) = -64 + 12 = -52. \end{aligned}$$

Final Answer 最終答案. -52.

Question 4 題目 4 Reciprocal substitution | 倒數代換

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

由 $ab = 1$ 先注意 $a, b \neq 0$，再寫成 $b = 1/a$。把含 b 的分式完全化為 a 的式子後，兩個分子正好相加成共同分母，因而不需要另行求出 a, b。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 倒數關係 $ab = 1$。
- 分式通分與互補結構。

Solution 解答

Since $b = 1/a$,

$$\frac{1}{1+b^2} = \frac{1}{1+1/a^2} = \frac{a^2}{1+a^2}.$$

Hence

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} = \frac{1}{1+a^2} + \frac{a^2}{1+a^2} = 1.$$

Final Answer 最終答案. 1.

Question 5 題目 5 Biquadratic equations 丨雙二次方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

令 $y = x^2 \geq 0$，先把四次式降為 y 的二次方程。解出兩個候選值後，利用 $y \geq 0$ 排除不合條件的根，最後由 $x^2 = y$ 還原所有實數 x，並檢查正負兩根都要保留。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 雙二次方程換元。
- 非負條件篩選根。

Solution 解答

Let $y = x^2 \geq 0$. Then

$$y^2 + y - \frac{5}{16} = 0,$$

so

$$16y^2 + 16y - 5 = (4y - 1)(4y + 5) = 0.$$

The nonnegative solution is $y = 1/4$. Hence

$$x^2 = \frac{1}{4}, \qquad x = \pm\frac{1}{2}.$$

Final Answer 最終答案. $x = \pm\frac{1}{2}$.

Question 6 題目 6 Sums of positive cubes 丨正整數立方和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先構造四個立方數的表示，證明 $n \leq 4$。再假設只用三個立方，按最大底數 $a = 9, 10, 11, 12$ 分情況；由 $b \geq c$ 可把第二大底數限制在極少數候選，逐一排除。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 立方數的大小估計。
- 有序化 $a \geq b \geq c$ 後縮窄搜尋範圍。
- 構造上界與排除下界。

Solution 解答

First,

$$2025 = 11^3 + 7^3 + 7^3 + 2^3,$$

so four cubes are sufficient.
We show that three cubes are impossible. Suppose

$$2025 = a^3 + b^3 + c^3, \qquad a \geq b \geq c \geq 1.$$

Since $3 \cdot 8^3 < 2025 < 13^3$, one has $9 \leq a \leq 12$. For a fixed a, the inequalities $b \geq c$ and $b^3 + c^3 = 2025 - a^3$ give

$$\sqrt[3]{\frac{2025 - a^3}{2}} \leq b \leq \sqrt[3]{2025 - a^3}.$$

This leaves only the following possibilities:

a	b	$c^3 = 2025 - a^3 - b^3$
12	6	81
11	8	182
10	10 or 9	25 or 296
9	9	567

None of $81, 182, 25, 296, 567$ is a positive cube. Thus three cubes are impossible.
A representation by two cubes would have $a \geq b \geq 1$. Since $2 \cdot 10^3 < 2025 < 2 \cdot 11^3$, one must have $a = 11$ or 12. The remaining values are

$$2025 - 11^3 = 694, \qquad 2025 - 12^3 = 297,$$

neither of which is a cube. A one-cube representation is also impossible because $12^3 < 2025 < 13^3$.
Hence the least number is 4.

Final Answer 最終答案. $n = 4$.

Question 7 題目 7 Nested radicals ｜嵌套根式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把前兩個根式分別設為 $u, v > 0$，計算 $(u+v)^2$。交叉項 $2uv$ 正好化成 $\sqrt{a^2-1}$，所以第三個嵌套根式就是 $\sqrt{(u+v)^2}$；再利用 $u+v>0$ 去掉絕對值，即可直接相消。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 共軛型根式。
- 先平方辨認嵌套根式。
- 正號條件的使用。

Solution 解答

Put

$$u = \sqrt{\frac{a+1}{2}}, \qquad v = \sqrt{\frac{a-1}{2}}.$$

Then $u, v > 0$ and

$$(u+v)^2 = u^2 + v^2 + 2uv = a + \sqrt{a^2-1}.$$

Therefore

$$\sqrt{a + \sqrt{a^2-1}} = u + v,$$

and the given expression is 0.

Final Answer 最終答案. 0.

Question 8 題目 8 Comparing large powers | 大冪比較

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

不直接計算大數，而先找小指數比較：$3^7 > 2^{11}$。立方後得到比所需更强的 $3^{21} > 2^{33}$。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 指數估值。
- 建立較强中間不等式。

Solution 解答

Since

$$3^7 = 2187 > 2048 = 2^{11},$$

cubing both sides gives

$$3^{21} > 2^{33} > 2^{31}.$$

Final Answer 最終答案. 3^{21} is larger.

Question 9 題目 9 Integer factorisation ｜整數因式分解

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把分式方程整理成 Simon's Favorite Factoring Trick：$(2a-19)(2b-19)=19^2$。因 $a,b\ge 10$，兩因子均為正，只需列出 361 的正因數配對。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 整數分式方程。
- 配方法式因式分解（SFFT）。
- 因數配對與極值。

Solution 解答

The equation is equivalent to

$$19(a+b) = 2ab.$$

Hence

$$(2a-19)(2b-19) = 361 = 19^2.$$

Also $a, b \ge 10$, so both factors are positive. The positive factor pairs are

$$(1, 361), \quad (19, 19), \quad (361, 1).$$

They give

$$(a, b) = (10, 190), \quad (19, 19), \quad (190, 10).$$

Thus the maximum difference is

$$190 - 10 = 180.$$

Final Answer 最終答案. 180.

Question 10 題目 10 Digit sums ︱數字和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

利用 $n \equiv s(n) \pmod 9$，把可能的數字和限制為 9 的倍數。又因 $n \le 2025$，數字和不超過 28，所以只需檢查 $9, 18, 27$ 三種情況。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 十進制數字和模 9 性質。
- 上界估計與有限檢查。

Solution 解答

Because $n \equiv s(n) \pmod 9$,

$$2025 = n + s(n) \equiv 2s(n) \pmod 9.$$

As $2025 \equiv 0 \pmod 9$, it follows that $s(n) \equiv 0 \pmod 9$. Moreover $n \le 2025$, so $s(n) \le 28$. Hence

$$s(n) \in \{9, 18, 27\}.$$

The corresponding candidates are

$$n = 2016, \quad 2007, \quad 1998.$$

Their digit sums are respectively

$$9, \quad 9, \quad 27.$$

Therefore the valid integers are 2016 and 1998.

Final Answer 最終答案. $n = 1998$ or $n = 2016$.

Question 11 題目 11 Tangential polygons ︱切圓多邊形

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

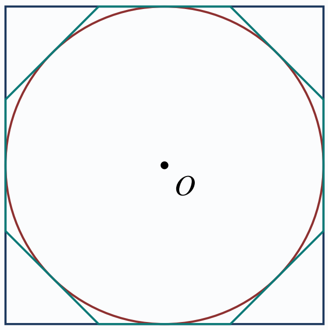
以圓心為基準觀察相鄰切線的法線夾角。若相鄰兩切線的法線夾角為 δ，由切線交點到兩切點的切線段長均為 $\tan(\delta/2)$。每個正方形角的 90° 被新增切線分成兩角，利用對稱或凸性可知兩角相等時貢獻最小。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 不等式 Inequalities

- 圓外兩切線與半角正切。
- 切圓多邊形周長公式 $P = 2\sum\tan(\delta_i/2)$。
- 對稱最優與凸函數。

Solution 解答



the minimizing regular tangential octagon

Figure 39. At the minimum, each right angle is split equally by the additional tangent. 最小值情況為正切八邊形的法線等角分布。

Let the additional tangent in one quadrant split the 90° angle between two adjacent square sides into α and $90^\circ - \alpha$. For a unit circle, the two tangent-length contributions in this quadrant are

$$\tan\frac{\alpha}{2} + \tan\left(45^\circ - \frac{\alpha}{2}\right).$$

The function $\tan x$ is strictly convex on $(0, 45^\circ)$, so the sum is minimized when the two arguments are equal:

$$\frac{\alpha}{2} = 45^\circ - \frac{\alpha}{2}, \qquad \alpha = 45^\circ.$$

For a tangential polygon, each tangent segment is counted twice in the perimeter. Therefore

$$P_{\min} = 2 \cdot 4 \cdot 2\tan 22.5^\circ = 16\tan 22.5^\circ.$$

Since $\tan 22.5^\circ = \sqrt{2} - 1$,

$$P_{\min} = 16(\sqrt{2} - 1).$$

Final Answer 最終答案. $\boxed{16(\sqrt{2} - 1)}$.

Question 12 題目 12 Cubic differences and integer bounds | 立方差與整數估計

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

因立方差為正，令 $d = a - b \geq 1$。把比值精確化成

$$\frac{a^3 - b^3}{ab} = 3d + \frac{d^3}{b(b+d)}.$$

由其大於 $3d$ 先限制 $d \leq 3$，再逐一處理三個小情況。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory | 代數 Algebra

- 立方差分解。
- 引入差值參數 $d = a - b$。
- 夾逼後的有限分類。

Solution 解答

We must have $a > b$. Put $d = a - b$, so $a = b + d$. Then

$$\frac{a^3 - b^3}{ab} = \frac{d(3b^2 + 3bd + d^2)}{b(b+d)} = 3d + \frac{d^3}{b(b+d)}.$$

The required inequality becomes

$$11 \leq 3d + \frac{d^3}{b(b+d)} \leq 12.$$

Since the fraction is positive, $3d < 12$, so $d \leq 3$.
If $d = 1$, the expression is less than 4; if $d = 2$, it is at most

$$6 + \frac{8}{1 \cdot 3} < 9.$$

Both are impossible. Thus $d = 3$, and

$$11 \leq 9 + \frac{27}{b(b+3)} \leq 12.$$

Hence

$$2 \leq \frac{27}{b(b+3)} \leq 3,$$

or

$$9 \leq b(b+3) \leq \frac{27}{2}.$$

The only positive integer satisfying this is $b = 2$. Therefore $a = b + 3 = 5$.

Final Answer 最終答案. $(a, b) = (5, 2)$.

Question 13 題目 13 Angle chasing in an equilateral triangle | 等邊三角形角度追蹤

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把邊長放大為 3。由 $CF = AD = 1$ 可得到 F 在 BC 上距 C 三分之一；坐標顯示 DF 與底邊成 $60°$，而 EF 垂直底邊。設 DC 與底邊的方向角為 θ，利用 D, E 關於中線對稱，兩所求角中的 θ 會相消。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 解析幾何 Analytic Geometry

- 等邊三角形坐標化。
- 方向角與對稱。
- $30°$–$60°$–$90°$ 結構。

Solution 解答

Scale the equilateral triangle so that $AB = 3$. Take

$$A = (0, 0), \quad B = (3, 0), \quad C = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right),$$

so

$$D = (1, 0), \qquad E = (2, 0).$$

Since $CF = 1$ and $CB = 3$, the point F is one third of the way from C to B:

$$F = (2, \sqrt{3}).$$

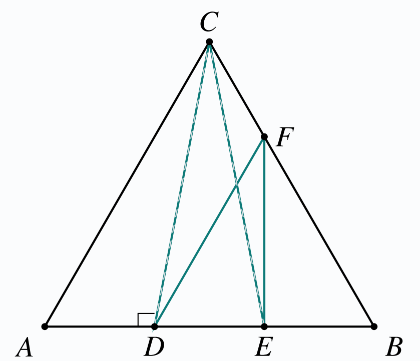


Figure 40. Coordinate model of the configuration. 題目配置的坐標模型。

The vector $\overrightarrow{DF} = (1, \sqrt{3})$ makes an angle 60° with the positive x-axis, while EF is vertical and makes an angle 90°. Let the direction angle of DC be θ. Since C lies on the perpendicular bisector of DE, the direction angle of EC is $180^\circ - \theta$. Therefore

$$\angle CDF = \theta - 60^\circ,$$

and

$$\angle CEF = (180^\circ - \theta) - 90^\circ = 90^\circ - \theta.$$

Thus

$$\angle CDF + \angle CEF = 30^\circ.$$

Final Answer 最終答案. 30°.

Question 14 題目 14 Vieta's formulas ｜韋達定理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把共同值設為 k，則 a, b, c 都是同一三次方程 $x^3 - kx - 1 = 0$ 的三個互異根。由韋達定理得到三個基本對稱式，再用立方和恆等式。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 由共同函數值構造多項式。
- 三次方程韋達定理。
- $a^3 + b^3 + c^3 - 3abc$ 恆等式。

Solution 解答

Let the common value be k. Then each of a, b, c satisfies

$$x^3 - kx - 1 = 0.$$

Because a, b, c are three distinct roots, Vieta's formulas give

$$a + b + c = 0, \qquad abc = 1.$$

Using

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),$$

we obtain

$$a^3 + b^3 + c^3 = 3abc = 3.$$

Final Answer 最終答案. 3.

Question 15 題目 15 Cyclic fractions and divisibility | 循環分式與整除性

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先證循環分式和嚴格介乎 1 與 3，故整數值只能是 2。再把「和減 2」因式分解，利用四數互異得到 $ac = bd$。最後以最大公因數參數化 a, b, c, d，把總和寫成兩個大於 1 的整數之積。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory **代數 Algebra**

- 循環分式估值。
- 差式因式分解。
- 互素參數化與合數證明。

Solution 解答

Set

$$S = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a}.$$

Write

$$x = \frac{b}{a}, \quad y = \frac{c}{b}, \quad z = \frac{d}{c}, \quad w = \frac{a}{d}.$$

Then $xyzw = 1$ and

$$S = \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} + \frac{1}{1+w}.$$

After putting the four terms over the common denominator, the numerator of $S - 1$ is

$$2 + 2(x + y + z + w) + (xy + xz + xw + yz + yw + zw) > 0.$$

Thus $S > 1$. Applying the same result to $1/x, 1/y, 1/z, 1/w$ gives $4 - S > 1$, so $S < 3$. Since S is an integer, $S = 2$.
Now

$$\begin{aligned} S - 2 &= (ac - bd)\left[\frac{1}{(a+b)(c+d)} - \frac{1}{(b+c)(d+a)}\right] \\ &= \frac{(ac-bd)(a-c)(b-d)}{(a+b)(b+c)(c+d)(d+a)}. \end{aligned}$$

Because the four integers are distinct, $a \neq c$ and $b \neq d$. Since $S - 2 = 0$, it follows that

$$ac = bd.$$

Let $g = \gcd(a, b)$ and write

$$a = gx, \qquad b = gy, \qquad \gcd(x, y) = 1.$$

From $ac = bd$ we have $xc = yd$, so $y \mid c$ and $x \mid d$. Hence for some positive integer h,

$$c = hy, \qquad d = hx.$$

Therefore

$$a + b + c + d = g(x + y) + h(x + y) = (g + h)(x + y).$$

Both factors are at least 2, so the sum is composite and hence not prime. □

Final Answer 最終答案. $a + b + c + d$ is composite.

Question 16 題目 16 Elementary optimisation and inequalities ｜初等最值與不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

第一部分不用微積分：令 $y = x^3$，把 $[x(1 - x^3)]^3$ 寫成 $y(1 - y)^3$，以四項 AM–GM 求最大值。第二部分把 a 消去，分別對 $\sqrt{b} - b$ 與 $\sqrt[4]{c} - c$ 套用平方配方及第一部分的最大值。

Olympiad Knowledge Points 奧數知識點

不等式 Inequalities　代數 Algebra

- AM–GM 求單變量最大值。
- 變量代換 $y = x^3$。
- 將多變量目標拆成兩個獨立上界。
- 等號條件同步檢查。

Solution 解答

For part (i), put $y = x^3 \in [0, 1]$. Then

$$(x - x^4)^3 = x^3(1 - x^3)^3 = y(1 - y)^3.$$

By AM–GM applied to

$$y, \quad \frac{1 - y}{3}, \quad \frac{1 - y}{3}, \quad \frac{1 - y}{3},$$

we get

$$y\left(\frac{1 - y}{3}\right)^3 \le \left(\frac{1}{4}\right)^4.$$

Thus
$$y(1-y)^3 \le \frac{27}{256},$$
and hence
$$x - x^4 \le \sqrt[3]{\frac{27}{256}} = \frac{3\sqrt[3]{2}}{8}.$$
Equality holds when $y = 1/4$, that is, $x = 1/\sqrt[3]{4}$.
For part (ii), since $a = 1 - b - c$,
$$a + \sqrt{b} + \sqrt[4]{c} = 1 + (\sqrt{b} - b) + (\sqrt[4]{c} - c).$$
Let $t = \sqrt{b}$. Then
$$t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \le \frac{1}{4}.$$
Let $r = \sqrt[4]{c}$. By part (i),
$$r - r^4 \le \frac{3\sqrt[3]{2}}{8}.$$
Therefore
$$a + \sqrt{b} + \sqrt[4]{c} \le 1 + \frac{1}{4} + \frac{3\sqrt[3]{2}}{8} = \frac{10 + 3\sqrt[3]{2}}{8}.$$
Equality is attained when
$$b = \frac{1}{4}, \qquad c = \frac{\sqrt[3]{2}}{8}, \qquad a = \frac{6 - \sqrt[3]{2}}{8}.$$

Final Answer 最終答案. $\max\limits_{0 \le x \le 1}(x - x^4) = \dfrac{3\sqrt[3]{2}}{8}$, and the stated inequality follows.

2024/2025 | Senior Division高中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Telescoping radicals 丨望遠鏡根式和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把每個分母有理化後，單項變成相鄰奇數平方根之差的一半。全部相加時中間項逐一抵消，只剩首尾兩項。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 望遠鏡級數。
- 根式有理化。
- 索引與首尾項管理。

Solution 解答

Rationalising the general term,

$$\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}} = \frac{\sqrt{2k+1}-\sqrt{2k-1}}{2}.$$

Therefore the sum telescopes:

$$\begin{aligned} S &= \frac{1}{2}\sum_{k=1}^{1012}\left(\sqrt{2k+1}-\sqrt{2k-1}\right) \\ &= \frac{1}{2}\left(\sqrt{2025}-\sqrt{1}\right) \\ &= \frac{1}{2}(45-1) = 22. \end{aligned}$$

Final Answer 最終答案. 22, so the correct choice is **A.**

Question 2 題目 2 Radicals with negative variables 丨含負變量的根式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先由根號內非負確定 $x \le 0$。令 $t = -x \ge 0$，便可按非負量正確抽出平方因子，避免錯把 $\sqrt{x^2}$ 寫成 x 而忽略絕對值。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 根式定義域。
- $\sqrt{x^2} = |x|$。
- 負變量代換。

Solution 解答

The radicand is nonnegative only when $x \le 0$. Put $t = -x \ge 0$. Then

$$\sqrt{-81x^3} = \sqrt{81t^3} = 9t\sqrt{t} = -9x\sqrt{-x}.$$

Final Answer 最終答案. $-9x\sqrt{-x}$, so the correct choice is **B**.

Question 3 題目 3 Logarithms and inequalities ｜對數與不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

令 $x = \log_{10} a$、$y = \log_{10} b$，原本的指數條件便轉成固定的 $x^2 + y^2$。目標是最大化 xy，直接由 $x^2 + y^2 \ge 2xy$ 得到上界，並檢查 $x = y$ 可達等號。

Olympiad Knowledge Points 奧數知識點

代數 Algebra　不等式 Inequalities

- 對數換元。
- 平方和與乘積：$(x - y)^2 \ge 0$。
- 等號條件與可達性。

Solution 解答

Set

$$x = \log_{10} a, \qquad y = \log_{10} b.$$

Then $a = 10^x$ and $b = 10^y$, so

$$a^{\log_{10} a} b^{\log_{10} b} = (10^x)^x (10^y)^y = 10^{x^2+y^2} = 16.$$

Hence

$$x^2 + y^2 = \log_{10} 16.$$

Since

$$x^2 + y^2 \geq 2xy,$$

we obtain

$$xy \leq \frac{1}{2} \log_{10} 16 = \log_{10} 4.$$

Equality occurs when $x = y$, and such positive a, b exist.

Final Answer 最終答案. $\max(\log_{10} a \log_{10} b) = \log_{10} 4$.

Question 4 題目 4 Cubic equations | 三次方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先試出簡單有理根 $x = \frac{1}{2}$，再把三次式分解為一次因子與二次因子。二次因子的判別式為負，因此不再產生實根。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 有理根試探。
- 多項式因式分解。
- 二次判別式。

Solution 解答

Move all terms to one side and factor:

$$x^3 + x - \frac{5}{8} = \frac{1}{8}(2x - 1)(4x^2 + 2x + 5).$$

The quadratic factor has discriminant

$$2^2 - 4(4)(5) = 4 - 80 < 0,$$

so it has no real roots. Therefore the only real solution comes from $2x - 1 = 0$.

Final Answer 最終答案. $x = \frac{1}{2}$.

Question 5 題目 5 Optimisation and inequality ｜最值與不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

第一部分把 $x - x^4$ 視作單變量函數求最大值。第二部分先用 $a = 1 - b - c$ 將目標拆成 $1 + (\sqrt{b} - b) + (\sqrt[4]{c} - c)$，兩個括號可分別獨立估值，最後把上界相加並核對等號條件。

Olympiad Knowledge Points 奧數知識點

不等式 Inequalities | 代數 Algebra

- 單變量極值（本稿採微分）。
- 拆項與分離變量。
- 配方法及等號條件。

Solution 解答

For part (i), let

$$f(x) = x - x^4.$$

Then

$$f'(x) = 1 - 4x^3, \qquad f''(x) = -12x^2 \le 0.$$

The unique critical point is

$$x = \frac{1}{\sqrt[3]{4}},$$

and it gives the maximum. Thus

$$\max f(x) = \frac{1}{\sqrt[3]{4}} - \frac{1}{4\sqrt[3]{4}} = \frac{3\sqrt[3]{2}}{8}.$$

For part (ii), use $a = 1 - b - c$:

$$a + \sqrt{b} + \sqrt[4]{c} = 1 + (\sqrt{b} - b) + (\sqrt[4]{c} - c).$$

Let $t = \sqrt{b}$. Then

$$t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \le \frac{1}{4}.$$

Let $r = \sqrt[4]{c}$. By part (i),

$$r - r^4 \le \frac{3\sqrt[3]{2}}{8}.$$

Therefore

$$a + \sqrt{b} + \sqrt[4]{c} \le 1 + \frac{1}{4} + \frac{3\sqrt[3]{2}}{8} = \frac{10 + 3\sqrt[3]{2}}{8}.$$

Equality occurs when

$$b = \frac{1}{4}, \qquad \sqrt[4]{c} = \frac{1}{\sqrt[3]{4}},$$

that is,

$$c = \frac{\sqrt[3]{2}}{8}, \qquad a = \frac{6 - \sqrt[3]{2}}{8}.$$

Final Answer 最終答案. $\max_{[0,1]}(x - x^4) = \frac{3\sqrt[3]{2}}{8}$, which yields the required inequality.

Question 6 題目 6 Coordinate geometry and reflection | 坐標幾何與反射

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

利用弧中點 D 得知 AD 為角平分線，故選 AD 為坐標軸最自然。算出外心與 D 的坐標，再把 X, M, N 逐一參數化；最後發現 $\overrightarrow{AM}$ 關於 AD 的反射方向與 $\overrightarrow{AN}$ 平行，角度結論隨即成立。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 解析幾何 Analytic Geometry

- 弧中點與角平分線。
- 外接圓方程。
- 向量參數式與平行條件。
- 軸反射保角。

Solution 解答

Since D is the midpoint of arc BC, AD bisects $\angle A$.

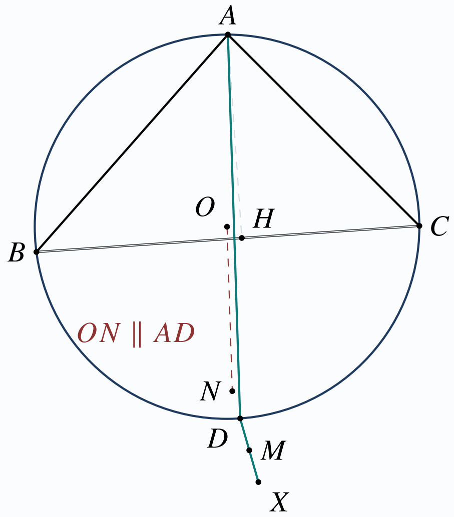

Figure 41. A more faithful sketch for Question 6. 第 6 題較精確示意圖。

Put $A = (0,0)$ and take AD as the x-axis. Let

$$\theta = \angle BAD = \angle DAC, \qquad u = \cos\theta, \qquad v = \sin\theta.$$

Write $AB = c$, $AC = b$, and set

$$s = b + c, \qquad d = b - c.$$

Then

$$B = (cu, cv), \qquad C = (bu, -bv).$$

The circumcircle through A has equation

$$x^2 + y^2 + \ell x + my = 0.$$

Substitution of B and C gives

$$\ell = -\frac{s}{2u}, \qquad m = \frac{d}{2v}.$$

Hence

$$O = \left(\frac{s}{4u}, -\frac{d}{4v}\right).$$

The second intersection of the x-axis with the circle is

$$D = \left(\frac{s}{2u}, 0\right).$$

Now

$$\overrightarrow{BC} = (du, -sv),$$

so a perpendicular direction is (sv, du). Since X lies on ray AH, write

$$X = \lambda(sv, du), \qquad \lambda > 0.$$

As M is the midpoint of DX,

$$\overrightarrow{AM} = \left(\frac{s(1 + 2\lambda uv)}{4u}, \frac{\lambda du}{2}\right).$$

Write $N = D + t(X - D)$. Since $ON \parallel AD$, the line ON is horizontal, so

$$y_N = y_O = -\frac{d}{4v}.$$

Because $y_N = t\lambda du$, we obtain

$$t = -\frac{1}{4\lambda uv}.$$

It follows that

$$\overrightarrow{AN} = \left(\frac{s(1+2\lambda uv)}{8\lambda u^2 v}, -\frac{d}{4v}\right).$$

Reflect $\overrightarrow{AM}$ in the x-axis. The reflected vector is

$$\left(\frac{s(1+2\lambda uv)}{4u}, -\frac{\lambda du}{2}\right) = 2\lambda uv\,\overrightarrow{AN}.$$

Thus ray AM is reflected onto ray AN by reflection in AD. The same reflection sends ray AB onto ray AC. Therefore

$$\angle BAM = \angle CAN.$$

□

Final Answer 最終答案. $\angle BAM = \angle CAN.$

Question 7 題目 7 Permutations and the triangle inequality ｜排列與三角不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把排列看成在數線上依次訪問 $1, 2, \ldots, n$ 的閉合路線。閉合路線包含兩條由 1 到 n 的路徑，每條長度至少為 $n-1$；再給出單調次序的排列達到下界。

Olympiad Knowledge Points 奧數知識點

組合 Combinatorics | 不等式 Inequalities

- 路徑模型。
- 三角不等式。
- 下界證明與構造達成。

Solution 解答

View the permutation as a closed route visiting all points $1, 2, \ldots, n$ on the real line.

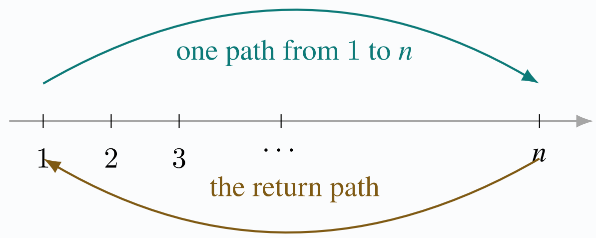

Figure 42. The cyclic order determines two routes joining 1 and n on the real line.

The closed route contains two distinct paths from 1 to n. By the triangle inequality, each path has length at least

$$n - 1.$$

Therefore

$$S \geq 2(n - 1).$$

Equality is attained by the cyclic ordering

$$1, 2, 3, \ldots, n,$$

for which

$$S = (n - 1) + (n - 1) = 2(n - 1).$$

Final Answer 最終答案. $\min S = 2(n - 1) = 2n - 2.$

Question 8 題目 8 Complex geometry | 複數幾何

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把外接圓設為單位圓，利用 $\bar{z} = 1/z$ 簡化反射與弦線方程。依次求出 P, E, F, Q, R，再令 $T = QR \cap EF$；由過 A, T 的弦求出 Y，最後證明它同時是直線 PK 與圓的第二交點，其中 K 為 EF 中點。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 複數幾何 Complex Geometry

- 單位圓複數坐標。
- 點關於弦的反射公式。
- 直線與圓的第二交點公式。
- 共點、共線及中點的代數化。

Solution 解答

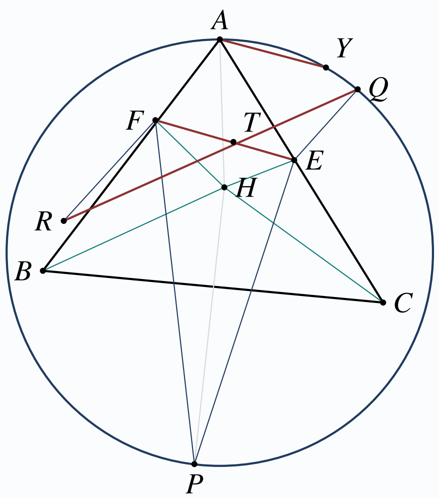

Figure 43. Lecture-note style schematic for Question 8. 第 8 題講義式示意圖。點 T 為 AY, QR, EF 的共點。

Use complex coordinates with Ω as the unit circle. Set

$$a = 1, \qquad |b| = |c| = 1, \qquad \sigma = b + c, \qquad \pi = bc.$$

Then

$$\bar{b} = \frac{1}{b}, \qquad \bar{c} = \frac{1}{c}, \qquad \bar{\sigma} = \frac{\sigma}{\pi}.$$

The orthocenter has coordinate

$$h = 1 + b + c.$$

Reflection in the chord BC sends z to $b + c - bc\bar{z}$. Since P is the reflection of H in BC,

$$p = b + c - bc\bar{h} = -bc = -\pi.$$

The reflection of B in AC is $1 + c - c\bar{b}$. Since E is the midpoint of B and this reflection,

$$e = \frac{1}{2}\left(b + 1 + c - \frac{c}{b}\right).$$

Similarly,

$$f = \frac{1}{2}\left(c + 1 + b - \frac{b}{c}\right).$$

The line EF has equation

$$z + \bar{z} = \mu, \qquad \mu = \frac{\pi\sigma - \sigma^2 + 4\pi + \sigma}{2\pi}.$$

Let K be the midpoint of EF. Its coordinate is

$$k = \frac{e + f}{2} = \frac{4\pi + 2\pi\sigma - \sigma^2}{4\pi},$$

and

$$\bar{k} = \frac{4\pi + 2\sigma - \sigma^2}{4\pi}.$$

For a point p on the unit circle and any point z, the second intersection of line pz with the unit circle is
$$\frac{z-p}{1-p\bar{z}}.$$
Hence the second intersection Y_0 of PK with Ω is
$$y_0=\frac{k-p}{1-p\bar{k}}=\frac{4\pi^2+2\pi\sigma+4\pi-\sigma^2}{\pi(4+4\pi+2\sigma-\sigma^2)}.$$
Likewise,
$$q=\frac{e-p}{1-p\bar{e}}=-\frac{2bc+b-c}{b(b-c-2)},$$
and
$$r=\frac{f-p}{1-p\bar{f}}=\frac{2bc-b+c}{c(b-c+2)}.$$
Put
$$v=qr,\qquad w=q+r.$$
Put the two fractions over a common denominator and use
$$(b-c)^2=(b+c)^2-4bc=\sigma^2-4\pi.$$
After replacing every occurrence of $b+c$ by σ and bc by π, the product and sum reduce to
$$v=\frac{\sigma^2-4\pi-4\pi^2}{\pi(\sigma^2-4\pi-4)},$$
$$w=\frac{-\sigma^3+2\sigma^2\pi+2\sigma^2-8\pi^2-8\pi}{\pi(\sigma^2-4\pi-4)}.$$
Since Q,R lie on the unit circle, the chord QR has equation
$$z+v\bar{z}=w.$$
Let $T=QR\cap EF$, with coordinate t. From
$$t+\bar{t}=\mu,\qquad t+v\bar{t}=w,$$
we obtain
$$t=\frac{w-\mu v}{1-v}.$$
The chord through $A=1$ and a point $Y=y$ on the unit circle has equation
$$z+y\bar{z}=1+y.$$
Since T lies on AY, substituting $z=t$ gives
$$y=\frac{t-1}{1-\bar{t}}.$$
Using $\bar{t}=\mu-t$ and the expressions above, this simplifies to
$$y=\frac{4\pi^2+2\pi\sigma+4\pi-\sigma^2}{\pi(4+4\pi+2\sigma-\sigma^2)}=y_0.$$
Thus $Y=Y_0$, so P,K,Y are collinear. Since K is the midpoint of EF, the line PY bisects EF. □

Final Answer 最終答案. PY passes through the midpoint of EF.

Question 9 題目 9 Factorials and modular arithmetic ｜階乘與模算術

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先用大小估計排除 $b \le 5$。當 $b \ge 6$ 時，模 16 將 a 限制在 4 或 5；再用模 11 排除 $b \ge 11$，使問題縮成有限個小範圍檢查，最後找出唯一四次冪。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 階乘的模整除性。
- 四次剩餘模 16、模 11。
- 先模篩選、後有限檢查的整數方程策略。

Solution 解答

If $b \le 5$, then

$$a! + b! \le 2 \cdot 5! = 240 < 2024 + c^4,$$

which is impossible. Hence $b \ge 6$, so

$$b! \equiv 0 \pmod{16}.$$

Because $2024 \equiv 8 \pmod{16}$ and every fourth power is congruent to 0 or 1 modulo 16,

$$a! \equiv 8 \quad \text{or} \quad 9 \pmod{16}.$$

Checking $a \le 5$ and noting that $a! \equiv 0 \pmod{16}$ for $a \ge 6$, we obtain

$$a \in \{4, 5\}.$$

If $a = 4$, then

$$c^4 = b! - 2000.$$

If $b \ge 11$, reduction modulo 11 gives

$$c^4 \equiv -2000 \equiv 2 \pmod{11},$$

which is impossible because the fourth-power residues modulo 11 are

$$0, 1, 3, 4, 5, 9.$$

Thus $6 \le b \le 10$. Direct comparison with consecutive fourth powers gives no solution:

b	$b! - 2000$
6	< 0
7	$3040 \in (7^4, 8^4)$
8	$38320 \in (13^4, 14^4)$
9	$360880 \in (24^4, 25^4)$
10	$3626800 \in (43^4, 44^4)$.

If $a = 5$, then

$$c^4 = b! - 1904.$$

For $b \ge 11$, modulo 11 gives

$$c^4 \equiv -1904 \equiv 10 \pmod{11},$$

again impossible. Hence $6 \le b \le 10$. Checking these five cases,

$$8! - 1904 = 40320 - 1904 = 38416 = 14^4,$$

while the other four values lie strictly between consecutive fourth powers. Therefore

$$(a, b, c) = (5, 8, 14)$$

is the unique solution.

Final Answer 最終答案. $(a, b, c) = (5, 8, 14)$.

Question 10 題目 10 Homogenisation and Schur's inequality | 齊次化與 Schur 不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先以 $p = a + b + c$ 齊次化，令 $x = a/p$ 等，使 $x + y + z = 1$。原條件給出 p^3 的表示，目標不等式遂化成一個只含 x, y, z 的四次對稱不等式；最後由四次 Schur 不等式完成。

Olympiad Knowledge Points 奧數知識點

不等式 **Inequalities**

- 齊次化與歸一化。
- 對稱多項式。
- Schur 不等式（四次形式）。
- 由條件消去尺度參數。

Solution 解答

Let

$$p = a + b + c > 0, \qquad x = \frac{a}{p}, \qquad y = \frac{b}{p}, \qquad z = \frac{c}{p}.$$

Then

$$x + y + z = 1$$

and the given condition becomes

$$p^3(x^3 + y^3 + z^3 - xyz) = 2.$$

The desired inequality is

$$p^4(x^4 + y^4 + z^4) \ge p,$$

or equivalently

$$p^3(x^4 + y^4 + z^4) \ge 1.$$

Using the expression for p^3, it is enough to prove

$$2(x^4 + y^4 + z^4) \ge x^3 + y^3 + z^3 - xyz.$$

Since $x + y + z = 1$, the difference between the two sides equals

$$\begin{aligned}&2(x^4 + y^4 + z^4) - (x^3 + y^3 + z^3) + xyz \\ &= x^2(x-y)(x-z) + y^2(y-z)(y-x) + z^2(z-x)(z-y).\end{aligned}$$

This is Schur's inequality of degree 4 and is nonnegative for $x, y, z \ge 0$. For completeness, assume without loss of generality that $x \ge y \ge z$. Then

$$x^2(x-z) \ge y^2(y-z),$$

so the first two terms combine to a nonnegative quantity, while

$$z^2(z-x)(z-y) \ge 0.$$

Therefore the required inequality holds. It follows that

$$p^3(x^4 + y^4 + z^4) \ge 1,$$

and hence

$$a^4 + b^4 + c^4 \ge a + b + c.$$

□

Final Answer 最終答案. $a^4 + b^4 + c^4 \ge a + b + c.$

2025/2026 | Junior Division初中組

This independent companion separates the official paper from the editorial layer: use the DSEDJ source for the question, then use the strategy, knowledge map, and worked solution here for study.
本獨立題解把官方原題與編輯層分開：先以教青局原卷核對題目，再利用本冊的策略、知識點與完整解答學習。

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Geometric series | 等比級數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先把首兩項的重複 2^1 分開，再把其餘部分辨認為公比為 2 的有限等比級數。這類選擇題最有效的做法，是先整理結構，再套用級數公式，避免逐項相加時漏項。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 有限等比級數：$1 + r + \cdots + r^m = \dfrac{r^{m+1} - 1}{r - 1}$。
- 指數規律及同類項整理。
- 選擇題中的快速驗算：估計量級及檢查首尾項。

Solution 解答

Using the geometric-series formula,

$$N = 2 + 2(1 + 2 + 4 + 8 + 16) = 2 + 2\left(\frac{2^5 - 1}{2 - 1}\right) = 2 + 62 = 64.$$

Final Answer 最終答案. $N = 64$, so the correct choice is **D.**

Question 2 題目 2 Floor functions and radicals | 取整函數與根式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

根式之差本身不易直接判斷取整值；將每項有理化後，分母隨 k 增大，便可找出由哪一項開始整體落在 $(0, 1)$。最後只需精確處理少量臨界項。

Olympiad Knowledge Points 奧數知識點

代數 Algebra　估值 Estimation

- 根式有理化：$\sqrt{x}-\sqrt{y}=\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$。
- 取整函數 $\lfloor x\rfloor$ 的區間判定。
- 單調性與臨界值估計。

Solution 解答

Write the sum as

$$N=\sum_{k=1}^{202}\left\lfloor\sqrt{10k+6}-\sqrt{10k-4}\right\rfloor.$$

Rationalising each difference gives

$$\sqrt{10k+6}-\sqrt{10k-4}=\frac{10}{\sqrt{10k+6}+\sqrt{10k-4}}.$$

For $k=1$,

$$1<\frac{10}{\sqrt{16}+\sqrt{6}}<2,$$

and for $k=2$,

$$1<\frac{10}{\sqrt{26}+\sqrt{16}}<2.$$

Hence the first two floor values are both 1.
For every $k\geq 3$,

$$\sqrt{10k+6}+\sqrt{10k-4}\geq\sqrt{36}+\sqrt{26}>10,$$

so the corresponding difference lies strictly between 0 and 1. All remaining floor values are therefore 0. Thus

$$N=1+1=2.$$

Final Answer 最終答案. $N=2$, so the correct choice is **A.**

Question 3 題目 3 Estimation and irrationality | 估值與無理數

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

比較根式時，以鄰近完全平方數建立上下界；證明無理性時，反設整個和為有理數，再平方並分離根式，將問題歸結為某個非完全平方數的平方根不可能為有理數。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 以完全平方數夾逼根式。

- 根式和的無理性證明。
- 反證法及平方後消去根式。

Solution 解答

Clearly $\alpha > 0$. Moreover,

$$\alpha = \sqrt{60} + \sqrt{61} > \sqrt{49} + \sqrt{49} = 14,$$

so statement B is false. On the other hand,

$$\alpha < \sqrt{64} + \sqrt{64} = 16,$$

so statement C is true.
To prove irrationality, suppose that α were rational. Then

$$\alpha^2 = 121 + 2\sqrt{3660}$$

would be rational, forcing $\sqrt{3660}$ to be rational. This is impossible because 3660 is not a perfect square. Hence α is irrational.

Final Answer 最終答案. The false statement is **B**.

Question 4 題目 4 Divisibility and averaging 丨整除與平均值

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先作多項式除法，把「分式為整數」轉化成 $n + 45 \mid 2025$。之後只須列出大於 45 的正因數，再還原所有 n 並計算平均值。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 整除條件與餘數：$n^2 = (n + 45)(n - 45) + 2025$。
- 質因數分解及正因數枚舉。
- 由除數條件反推整數解。

Solution 解答

Polynomial division gives

$$n^2 = (n + 45)(n - 45) + 2025,$$

so

$$\frac{n^2}{n+45} = n - 45 + \frac{2025}{n+45}.$$

Therefore $n + 45$ must be a positive divisor of

$$2025 = 3^4 \cdot 5^2.$$

Since $n > 0$, we require $n + 45 > 45$. The divisors of 2025 greater than 45 are

$$75, 81, 135, 225, 405, 675, 2025.$$

Hence

$$n = 30, 36, 90, 180, 360, 630, 1980.$$

Their sum is

$$30 + 36 + 90 + 180 + 360 + 630 + 1980 = 3306,$$

and so

$$N = \frac{3306}{7}.$$

Final Answer 最終答案. $N = \dfrac{3306}{7}$, which is none of the listed choices.

Question 5 題目 5 Real algebraic systems | 實代數方程組

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

利用恆等式把兩個方程合併成 $(x^2+y^2)^2 = a^2+b^2$，先求出 x^2, y^2，再以 $2xy = b$ 決定符號。最後必須把 $b = 0$ 的退化情況獨立分類。

Olympiad Knowledge Points 奧數知識點

代數 Algebra | 複數觀點 Complex View

- 恆等式 $(x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2$。
- 複數平方 $(x+iy)^2 = a+ib$ 的坐標意義。
- 平方根的符號配對與分類討論。

Solution 解答

Observe that

$$a^2 + b^2 = (x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2.$$

Let

$$R = \sqrt{a^2+b^2} \geq 0.$$

Then

$$x^2 + y^2 = R, \qquad x^2 - y^2 = a,$$

so

$$x^2 = \frac{R+a}{2}, \qquad y^2 = \frac{R-a}{2}.$$

The signs must also satisfy $2xy = b$.
If $b \neq 0$, then $R > |a|$, and the two solutions are

$$(x, y) = \left(\pm\sqrt{\frac{R+a}{2}},\ \pm\frac{b}{\sqrt{2(R+a)}} \right),$$

where the same sign is taken in both coordinates.
If $b = 0$, the possibilities are:

condition	solutions
$a > 0$	$(\pm\sqrt{a}, 0)$
$a < 0$	$(0, \pm\sqrt{-a})$
$a = 0$	$(0, 0)$.

Final Answer 最終答案. The solutions are exactly the pairs listed above, with $R = \sqrt{a^2 + b^2}$.

Question 6 題目 6 Rational parametrisation | 有理參數表示

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

圓上已知一個有理點 $(1, 1)$，便可用過該點、斜率為有理數的直線截圓。代入後其中一根對應原點，另一根自然給出有理函數參數式，因此得到無窮多個有理解。

Olympiad Knowledge Points 奧數知識點

代數 Algebra | 解析幾何 Analytic Geometry

- 圓錐曲線的有理參數化。
- 已知有理點的割線法。
- 二次方程已知一根後求另一根。

Editorial Note 編按

The Chinese statement and the handwritten solution use *rational numbers*. The English line in the printed paper says “integers”, which would make the assertion false.

Solution 解答

The rational point $(1,1)$ lies on the circle $a^2+b^2=2$. Draw a line of rational slope t through $(1,1)$:

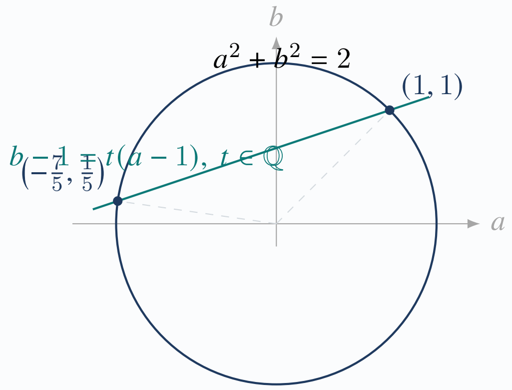

Figure 44. A rational-slope secant through $(1,1)$ meets the circle again at another rational point. 取有理斜率直線與圓再交，可產生新的有理點。

$$b-1=t(a-1),\qquad t\in\mathbb{Q}.$$

Put $u=a-1$, so $a=1+u$ and $b=1+tu$. Substitution into the circle gives

$$(1+u)^2+(1+tu)^2=2,$$

which simplifies to

$$u\big(2(1+t)+u(1+t^2)\big)=0.$$

The root $u=0$ corresponds to the original point $(1,1)$. The second intersection is obtained from

$$u=-\frac{2(1+t)}{1+t^2}.$$

Hence

$$a=\frac{t^2-2t-1}{t^2+1},\qquad b=\frac{1-2t-t^2}{t^2+1}.$$

For every rational $t\neq -1$, this gives a rational point on the circle distinct from $(1,1)$. Since there are infinitely many rational values of t, there are infinitely many rational solutions.

Final Answer 最終答案. One parametrisation is $(a,b)=\left(\frac{t^2-2t-1}{t^2+1},\frac{1-2t-t^2}{t^2+1}\right)$ for $t\in\mathbb{Q}$.

Question 7 題目 7 Modular arithmetic | 模算術

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

區間內只有五個奇數候選。對每個候選檢查 $2^{48} \equiv 1 \pmod{n}$；可利用較小次方的循環或分解指數，避免直接計算巨大整數。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory

- 模冪運算及快速降冪。
- 乘法階與指數週期。
- 在狹窄區間內以候選檢驗代替完整因數分解。

Solution 解答

Since $2^{48} - 1$ is odd, the only candidates are

$$61, 63, 65, 67, 69.$$

For 63,

$$2^6 \equiv 1 \pmod{63},$$

so $2^{48} \equiv 1 \pmod{63}$. For 65,

$$2^{12} = 4096 \equiv 1 \pmod{65},$$

so $2^{48} \equiv 1 \pmod{65}$.
The remaining candidates fail:

$$\begin{aligned} 2^{48} &\equiv 34 \pmod{61}, \\ 2^{48} &\equiv 62 \pmod{67}, \\ 2^{48} &\equiv 16 \pmod{69}. \end{aligned}$$

Thus the only divisors in the required interval are 63 and 65.

Final Answer 最終答案. $N = 63 + 65 = 128.$

Question 8 題目 8 A family of quadratics | 二次函數族

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

固定點部分先把式子整理成「不含參數的部分 + 參數乘上一個因子」，令該因子為零即可。根的範圍部分則把較大根 r 視作參數，反解 a，再用韋達定理寫出另一根 s，最後由 $r > s$ 化成有理不等式。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 含參數二次函數的固定點。
- 韋達定理。
- 有理不等式的符號表分析。
- 必要性與充分性的雙向檢查。

Solution 解答

We first rewrite the expression as

$$y = (x+1)^2 + a(x-2).$$

Therefore, when $x = 2$, the term involving a vanishes and

$$y = 3^2 = 9.$$

Thus every parabola passes through the fixed point $(2, 9)$.
For part (b), let r be the larger root and s the other root. Since r is a root,

$$(r+1)^2 + a(r-2) = 0,$$

so, for $r \neq 2$,

$$a = -\frac{(r+1)^2}{r-2}.$$

By Vieta's formula,

$$r + s = -(a+2),$$

and hence

$$s = -a - 2 - r = \frac{2r+5}{r-2}.$$

The condition that r is the larger root is $r > s$, giving

$$r - \frac{2r+5}{r-2} > 0.$$

Equivalently,

$$\frac{(r-5)(r+1)}{r-2} > 0.$$

A sign analysis yields

$$-1 < r < 2 \qquad \text{or} \qquad r > 5.$$

Conversely, every r in these intervals determines a real a and a distinct smaller root s, so the range is exact.

Final Answer 最終答案. The fixed point is $(2, 9)$, and the larger root satisfies $\boxed{-1 < r < 2 \text{ or } r > 5}$.

Question 9 題目 9 Binary representation and Vieta's formula | 二進表示與韋達定理

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先把兩集合的元素和相加，得到固定總和。若方程有正整數根，另一根亦為整數；用韋達定理把條件改寫成 $(r+1)(s+1)=2047$。因數對只有少數，再利用二進表示的唯一性確定集合。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory　組合 Combinatorics

- 韋達定理與整數根。
- 因式變形 $rs+r+s+1=(r+1)(s+1)$。
- 二進制表示的唯一性。
- 集合分割與有序分割。

Solution 解答

The total sum is
$$S(A)+S(B)=2^1+2^2+\cdots+2^{10}=2^{11}-2=2046.$$
Suppose one root is a positive integer r. Since the sum of the roots is the integer $S(A)$, the other root $s=S(A)-r$ is also an integer. Moreover,
$$r+s=S(A),\qquad rs=S(B).$$
Therefore
$$r+s+rs=2046,$$
so
$$(r+1)(s+1)=2047=23\cdot 89.$$
As $r>0$ and $rs=S(B)\geq 0$, we have $s\geq 0$. The possible unordered factor pairs are
$$(r+1,s+1)=(1,2047)\quad\text{or}\quad(23,89).$$
These give the root pairs
$$(r,s)=(0,2046)\quad\text{or}\quad(22,88).$$
In the first case,
$$S(A)=2046,\qquad S(B)=0,$$
so A contains all ten powers and $B=\varnothing$.
In the second case,
$$S(A)=22+88=110,\qquad S(B)=22\cdot 88=1936.$$
The binary expansion
$$110=64+32+8+4+2=2^6+2^5+2^3+2^2+2^1$$
uniquely determines A, with B its complement. Thus there are exactly two ordered divisions.

Final Answer 最終答案. 2 ordered divisions.

Question 10 題目 10 Coordinate geometry on a circle | 圓上的坐標幾何

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先把外接圓縮放為單位圓，並利用 M 是弧 AC 中點建立對稱坐標。再以角平分線、內切圓半徑及反射求出 I, J；求直線 MJ 與圓的另一交點 D，由 DO 為直徑方向得 $E = -D$，最後比較向量證明平行。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 解析幾何 Analytic Geometry

- 角平分線與對弧中點。
- 圓的坐標化及單位圓規範化。
- 點關於直線的反射。
- 向量平行判定與圓上第二交點。

Solution 解答

Scale the circumcircle to the unit circle and place

$$O = (0, 0), \qquad M = (0, -1).$$

Let $\angle ABC = \theta$. Since BM is the internal angle bisector, M is the midpoint of arc AC not containing B, so we may write

$$A = (-\sin\theta, -\cos\theta), \qquad C = (\sin\theta, -\cos\theta).$$

Thus AC is the horizontal line $y = -\cos\theta$.

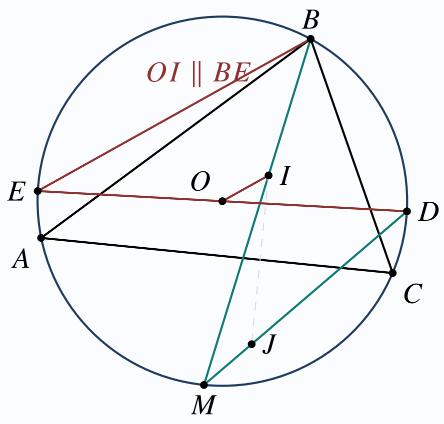

Figure 45. A more faithful geometric sketch for Question 10. 第 10 題較精確示意圖。

Let

$$B = (p, q), \qquad p^2 + q^2 = 1,$$

and define

$$t = \sqrt{1 - \cos\theta}, \qquad u = \sqrt{1 + q}, \qquad v = \frac{p}{u}.$$

Then

$$p = uv, \qquad q = u^2 - 1, \qquad u^2 + v^2 = 2.$$

Because B, I, M are collinear, write

$$I = M + \lambda(B - M) = (\lambda p, -1 + \lambda(q + 1)).$$

Let r be the inradius. Since $BI = (1 - \lambda)BM$,

$$BM = \sqrt{p^2 + (q + 1)^2} = \sqrt{2(1 + q)} = \sqrt{2}\,u$$

and

$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}} = \frac{t}{\sqrt{2}},$$

we obtain

$$r = (1 - \lambda)tu.$$

On the other hand, the vertical distance from I to AC is

$$r = -1 + \lambda(q + 1) + \cos\theta = \lambda u^2 - t^2.$$

Equating the two expressions gives

$$(1 - \lambda)tu = \lambda u^2 - t^2, \qquad \text{hence} \qquad \lambda = \frac{t}{u}.$$

Therefore

$$I = (tv, tu - 1).$$

Since AC is $y = t^2 - 1$, reflecting I in AC gives

$$J = (tv, 2t^2 - tu - 1).$$

Consequently,

$$\overrightarrow{MJ} = t(v, 2t - u).$$

A general point of MJ is

$$X = (\mu v, -1 + \mu(2t - u)).$$

The intersections with the unit circle satisfy

$$\mu^2 v^2 + [-1 + \mu(2t - v)]^2 = 1.$$

Besides the root $\mu = 0$ corresponding to M, the other root is

$$\mu = \frac{2(2t - u)}{v^2 + (2t - u)^2}.$$

Using $v^2 = 2 - u^2$, this gives

$$D_x = \frac{v(2t - u)}{1 + 2t^2 - 2tu}, \qquad D_y = \frac{-1 + 2t^2 - 2tu + u^2}{1 + 2t^2 - 2tu}.$$

Because DO is a diameter, E is antipodal to D, so $E = -D$. Hence

$$\overrightarrow{BE} = E - B = -(D + B).$$

A direct simplification yields

$$D + B = \frac{2(1 + tu - u^2)}{1 + 2t^2 - 2tu}\,(tv, tv - 1).$$

The vector on the right is a scalar multiple of $\overrightarrow{OI} = I$. Therefore $D + B$ is parallel to OI, and so is

$$\overrightarrow{BE} = -(D + B).$$

Thus

$$OI \parallel BE.$$

□

Final Answer 最終答案. $OI \parallel BE$.

2025/2026 | Senior Division高中組

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Independent Worked Solutions 獨立完整解答

Question 1 題目 1 Floor functions and radicals 丨取整函數與根式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先有理化每個根式差，再判斷其大小是否跨越整數 1。由於分母單調增大，只需檢查最前面的臨界項，其餘各項取整後均為零。

Olympiad Knowledge Points 奧數知識點

代數 Algebra　估值 Estimation

- 根式有理化。
- 取整函數的區間判斷。
- 單調性與臨界項分析。

Solution 解答

As in the Junior Division problem, write

$$\sum_{k=1}^{202} \left\lfloor \sqrt{10k+6} - \sqrt{10k-4} \right\rfloor .$$

Rationalisation gives

$$\sqrt{10k+6} - \sqrt{10k-4} = \frac{10}{\sqrt{10k+6} + \sqrt{10k-4}}.$$

For $k = 1, 2$ this number lies in $(1, 2)$, whereas for every $k \geq 3$ it lies in $(0, 1)$. Hence only the first two terms contribute.

Final Answer 最終答案. 2, so the correct choice is **A.**

Question 2 題目 2 Integer equations 丨整數方程

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

將立方差分解為兩個正整數因子。因 61 為質數，因子配對極少；排除不可能的配對後，令 $b - a = 1$，問題降為一個二次整數方程。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory | 代數 Algebra

- 立方差公式 $b^3 - a^3 = (b - a)(b^2 + ab + a^2)$。
- 質數的因子配對。
- 整數方程的篩選與回代。

Solution 解答

Rearranging and factorising,

$$(b - a)(b^2 + ab + a^2) = 61.$$

Since $61 > 0$, we have $b > a$. The second factor is positive, and 61 is prime. Thus

$$b - a = 1, \qquad b^2 + ab + a^2 = 61;$$

the alternative $b - a = 61$ and $b^2 + ab + a^2 = 1$ is impossible.
Putting $b = a + 1$ gives

$$(a + 1)^3 - a^3 = 61,$$

so

$$3a^2 + 3a + 1 = 61.$$

Therefore

$$a^2 + a - 20 = 0, \qquad (a + 5)(a - 4) = 0.$$

The two solutions are

$$(a, b) = (-5, -4), \qquad (4, 5).$$

Final Answer 最終答案. $N = 2$, so the correct choice is **C**.

Question 3 題目 3 Combinatorics and recurrence | 組合與遞推

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

根據字串末端狀態分類：以 A 結尾、以單個 B 結尾、以 BB 結尾。新增一個字母時，合法轉移完全由末端狀態決定，因此建立三個遞推量並逐步算到長度 9。

Olympiad Knowledge Points 奧數知識點

組合 Combinatorics

- 有限狀態遞推／動態規劃。
- 禁用子字串的計數。
- 按末端結構分類以避免重複或遺漏。

Solution 解答

Let

$$\begin{aligned} u_n &= \text{number of valid length-}n\text{ strings ending in }A,\\ v_n &= \text{number ending in exactly one trailing }B,\\ w_n &= \text{number ending in }BB. \end{aligned}$$

Appending one letter gives the recurrences

$$u_{n+1} = v_n + w_n, \qquad v_{n+1} = u_n, \qquad w_{n+1} = v_n.$$

For $n = 1$,

$$(u_1, v_1, w_1) = (1, 1, 0).$$

Successive values are

n	(u_n, v_n, w_n)
1	$(1, 1, 0)$
2	$(1, 1, 1)$
3	$(2, 1, 1)$
4	$(2, 2, 1)$
5	$(3, 2, 2)$
6	$(4, 3, 2)$
7	$(5, 4, 3)$
8	$(7, 5, 4)$
9	$(9, 7, 5)$.

Thus the required number is

$$u_9 + v_9 + w_9 = 9 + 7 + 5 = 21.$$

Final Answer 最終答案. 21, which is none of the listed choices.

Question 4 題目 4 Quadratic polynomials | 二次多項式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把給定根式根與其共軛根配成一對。兩根的和與積均為整數，直接用韋達形式 $(x-\alpha)(x-\beta)$ 構造所求首一二次多項式。

Olympiad Knowledge Points 奧數知識點

代數 Algebra

- 根式共軛。
- 由根構造多項式。
- 最小多項式的基本思想。

Solution 解答

Since $\sqrt{2025} = 45$, let

$$\alpha = \sqrt{2026} + 45.$$

Choose its algebraic conjugate

$$\beta = 45 - \sqrt{2026}.$$

Then

$$\alpha + \beta = 90, \qquad \alpha\beta = 45^2 - 2026 = -1.$$

Therefore the required polynomial is

$$(x - \alpha)(x - \beta) = x^2 - 90x - 1.$$

Final Answer 最終答案. $p(x) = x^2 - 90x - 1$, so the correct choice is **D**.

Question 5 題目 5 Finite sets and translations | 有限集合與平移

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

把條件解讀為集合包含關係 $A \cup B \subseteq (A - 17) \cup (B + 31)$。兩邊基數上界恰好相同，因此所有不等式必為等號，進而得到集合相等且右側不交。最後比較兩邊元素總和，平移所造成的總差便給出 $17n = 31m$。

Olympiad Knowledge Points 奧數知識點

組合 Combinatorics　數論 Number Theory

- 有限集合的平移不改變基數。
- 基數夾逼與等號條件。
- 對集合元素求和的雙重計數思想。

Solution 解答

Define the translated sets

$$A-17=\{a-17:a\in A\},\qquad B+31=\{b+31:b\in B\}.$$

For each $k\in A\cup B$, either $k+17\in A$, in which case $k\in A-17$, or $k-31\in B$, in which case $k\in B+31$. Hence

$$A\cup B\subseteq(A-17)\cup(B+31).$$

Translations preserve cardinality, so

$$|A-17|=n,\qquad |B+31|=m.$$

Since A and B are disjoint,

$$|A\cup B|=n+m.$$

Consequently,

$$n+m=|A\cup B|\le|(A-17)\cup(B+31)|\le|A-17|+|B+31|=n+m.$$

All inequalities are equalities. Therefore

$$A\cup B=(A-17)\cup(B+31),$$

and the two sets on the right are disjoint.
Summing all elements on both sides gives

$$\sum_{a\in A}a+\sum_{b\in B}b=\sum_{a\in A}(a-17)+\sum_{b\in B}(b+31).$$

Cancelling the common sums yields

$$-17n+31m=0.$$

Thus $17n=31m$. □

Final Answer 最終答案. $17n=31m$.

Question 6 題目 6 Mathematical games and sums of squares | 數學遊戲與平方和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

以「表示成至多 r 個平方和」作為單調量。Lagrange 四平方定理保證起點在 S_4；Amy 的乘冪不增加所需平方數，而 Bob 每次可減去表示中的一個非零平方，使層級下降一級，最終到達零。

Olympiad Knowledge Points 奧數知識點

數論 Number Theory | 博弈 Combinatorial Games

- Lagrange 四平方定理。
- 不變量／單調量（monovariant）。
- 構造性必勝策略。

Solution 解答

For $r \geq 1$, let S_r be the set of positive integers representable as a sum of at most r integer squares. By Lagrange's four-square theorem, every positive integer belongs to S_4.
Suppose the current number is

$$m = y_1^2 + \cdots + y_r^2 \in S_r.$$

Amy's operation cannot increase the number of squares needed. Indeed:
If $k = 2t$ is even, then

$$m^k = (m^t)^2 \in S_1 \subseteq S_r.$$

If $k = 2t + 1$ is odd, then

$$m^k = m(m^t)^2 = (m^t y_1)^2 + \cdots + (m^t y_r)^2 \in S_r.$$

Now suppose the current number lies in S_r. Choose a representation with at most r squares and let Bob subtract one nonzero square appearing in that representation. The resulting number lies in S_{r-1}. Amy's subsequent power move keeps it in S_{r-1}.
Starting from S_4, Bob can therefore force the sequence

$$S_4 \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1 \longrightarrow 0.$$

Thus Bob has a winning strategy in at most four of his moves.

Final Answer 最終答案. No. Amy cannot prevent Bob from winning.

Question 7 題目 7 Parity and prefix sums | 奇偶性與前綴和

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

將白、黑硬幣編碼為 $+1, -1$，把局部計數轉換成前綴和。好硬幣條件等價於步長為 1 的折線跨越半整數水平線 $T/2$；路徑起點與終點分居兩側，所以跨越次數必為奇數。

Olympiad Knowledge Points 奧數知識點

組合 Combinatorics

- 前綴和編碼。
- 奇偶性與路徑跨越。
- 將計數條件幾何化為格路問題。

Solution 解答

Encode the colour of the ith coin by

$$\varepsilon_i = \begin{cases} 1, & \text{if the coin is white,} \\ -1, & \text{if the coin is black.} \end{cases}$$

Define the prefix sums

$$P_i = \varepsilon_1 + \cdots + \varepsilon_i, \qquad P_0 = 0,$$

and let

$$T = P_{2n+1}.$$

For the ith coin, the number of white coins to its left is

$$\frac{(i-1) + P_{i-1}}{2}.$$

There are $2n+1-i$ coins to its right, whose signed sum is $T - P_i$. Hence the number of black coins to its right is

$$\frac{(2n+1-i) - (T - P_i)}{2}.$$

Their sum is

$$n + \frac{P_{i-1} + P_i - T}{2}.$$

Therefore the ith coin is good exactly when

$$P_{i-1} + P_i = T.$$

Since $P_i - P_{i-1} = \pm 1$, this condition says that the unit segment joining P_{i-1} to P_i crosses the level $T/2$.
The walk starts at $P_0 = 0$ and ends at $P_{2n+1} = T$. Because $2n+1$ is odd, T is odd, so neither endpoint lies on the half-integer level $T/2$. Moreover, 0 and T lie on opposite sides of this level. Every crossing switches sides. Hence the total number of crossings, and therefore the number of good coins, is odd.

Final Answer 最終答案. The number of good coins is always **odd.**

Question 8 題目 8 Symmetric inequalities 丨對稱不等式

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先對任意一對 x, y 建立可精確分解為四次方的局部不等式，再對六對變量求和。對稱性使平方項出現次數一致，配合 $\sum x_i = 1$ 消去所有依賴項；等號則要求每一對變量相等。

Olympiad Knowledge Points 奧數知識點

不等式 Inequalities

- 局部配對不等式。
- 對稱和與 $\sum_{i<j} x_i x_j$ 的化簡。
- 等號條件的同步追蹤。

Solution 解答

For any $x, y \geq 0$,

$$\frac{1}{4}(x^2+y^2)+\frac{3}{2}xy-(x+y)\sqrt{xy}=\frac{1}{4}(\sqrt{x}-\sqrt{y})^4 \geq 0.$$

Thus

$$(x+y)\sqrt{xy} \leq \frac{1}{4}(x^2+y^2)+\frac{3}{2}xy.$$

Applying this to all six pairs gives

$$S \leq \frac{1}{4}\sum_{i<j}(x_i^2+x_j^2)+\frac{3}{2}\sum_{i<j}x_i x_j.$$

Each x_i^2 appears in exactly three pairs, so

$$\sum_{i<j}(x_i^2+x_j^2)=3\sum_{i=1}^{4}x_i^2.$$

Also,

$$\sum_{i<j}x_i x_j=\frac{(x_1+x_2+x_3+x_4)^2-\sum x_i^2}{2}=\frac{1-\sum x_i^2}{2}.$$

Therefore

$$S \leq \frac{3}{4}\sum x_i^2+\frac{3}{4}\left(1-\sum x_i^2\right)=\frac{3}{4}.$$

Equality in the pairwise inequality requires $x_i = x_j$ for every pair, hence

$$x_1=x_2=x_3=x_4=\frac{1}{4}.$$

At these values, $S = 3/4$.

Final Answer 最終答案. $\max S = \frac{3}{4}$, attained at $x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$.

Question 9 題目 9 Euclidean and coordinate geometry | 歐氏與坐標幾何

Problem locator / 題目定位： consult the official DSEDJ paper for this division and question number before reading the solution.
請先按本組別及題號查閱教青局官方原卷，再閱讀下列策略與解答。

Solution Strategy 思路解析

先用旁切圓切線長關係得到 $AK = AL = s$，再把角平分線設為坐標軸，使 K, L 對稱。中點 N 落在外接圓給出一個邊長恆等式；第二套坐標中寫出 I, M, O，以三點面積行列式檢驗共線，並用前述恆等式令行列式為零。

Olympiad Knowledge Points 奧數知識點

幾何 Geometry | 解析幾何 Analytic Geometry

- 旁切圓與等切線長。
- 角平分線坐標系。
- 圓方程及弦中點條件。
- 三點共線的行列式判定。

Solution 解答

Let

$$a = BC, \qquad b = CA, \qquad c = AB, \qquad s = \frac{a+b+c}{2}.$$

Let N be the midpoint of KL.

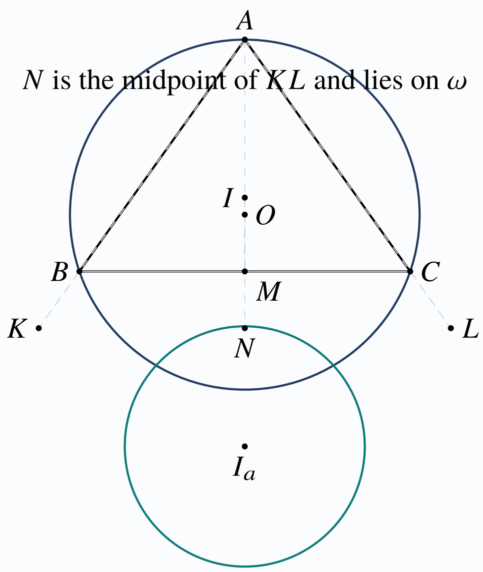

Figure 46. Schematic configuration for the A-excircle problem. 圖中標示 A-旁心圓、外接圓及中點 N。

By equal tangent lengths from A,

$$AK = AL = s.$$

Place A at the origin and take the internal angle bisector of $\angle A$ as the x-axis. Write $\theta = A/2$. We may take

$$B = (c\cos\theta, c\sin\theta), \qquad C = (b\cos\theta, -b\sin\theta).$$

Thus

$$K = (s\cos\theta, s\sin\theta), \qquad L = (s\cos\theta, -s\sin\theta),$$

so

$$N = (s\cos\theta, 0).$$

Write the circumcircle in the form

$$x^2 + y^2 - \lambda x - \mu y = 0.$$

Substituting B and C gives

$$\lambda\cos\theta + \mu\sin\theta = c, \qquad \lambda\cos\theta - \mu\sin\theta = b,$$

so

$$\lambda = \frac{b+c}{2\cos\theta}.$$

The intersections of the circumcircle with the x-axis are $x = 0$ and $x = \lambda$. Since $N \neq A$ and N lies on the circumcircle,

$$s\cos\theta = \lambda = \frac{b+c}{2\cos\theta}.$$

Hence

$$2s\cos^2\theta = b + c.$$

Using $2\cos^2\theta = 1 + \cos A$ and the cosine law,

$$s\left(1 + \frac{b^2+c^2-a^2}{2bc}\right) = b + c.$$

After clearing denominators and using $2s = a + b + c$, this becomes

$$F(a, b, c) = 0,$$

where

$$F(a, b, c) = a^3 + a^2 b + a^2 c - ab^2 - 2abc - ac^2 - b^3 + b^2 c + bc^2 - c^3.$$

We now use a second coordinate system. Put

$$B = (0, 0), \qquad C = (a, 0), \qquad A = (p, q),$$

where

$$p = \frac{a^2 + c^2 - b^2}{2a}, \qquad p^2 + q^2 = c^2.$$

Because the A-excircle touches BC at M and $BM = s - c$,

$$M = (s - c, 0).$$

The incenter has barycentric coordinates $a : b : c$, hence

$$I = \left(\frac{a(p + c)}{2s}, \frac{aq}{2s} \right).$$

The circumcenter lies on the perpendicular bisector of BC, so its x-coordinate is $a/2$. From $OA = OB$,

$$O = \left(\frac{a}{2}, \frac{c^2 - ap}{2q} \right).$$

The signed-area determinant of I, M, O is

$$\Delta = \det \begin{pmatrix} I_x - M_x & I_y - M_y \\ O_x - M_x & O_y - M_y \end{pmatrix}.$$

Substituting the coordinates and simplifying with

$$p = \frac{a^2 + c^2 - b^2}{2a}, \qquad q^2 = c^2 - p^2,$$

gives

$$\Delta = \frac{b - c}{8aq} F(a, b, c).$$

But $F(a, b, c) = 0$ from the midpoint condition. Therefore $\Delta = 0$, proving that I, M, O are collinear. □

Final Answer 最終答案. I, M, O are collinear.

Consolidated Answer Key | 歷屆答案索引

Editorial Note 編按

This appendix is a navigation aid. Where an original question is ambiguous, the answer below retains the qualification stated in the full solution; consult the linked question for the complete argument. 本附錄只供快速查閱。如原題存在歧義，答案會保留完整解答中的限制說明；請按題號返回正文閱讀論證。

2019 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Prime distribution ｜質數分布	$(x, y) = (4, 0)$, so the correct choice is **E**.
2	Ratios of fractions ｜分數比例	$\frac{1}{17}$, which is none of the listed choices.
3	Digit sum ｜數字和	18163, so the correct choice is **C**.
4	Integer bounds and ratios ｜整數界與比值	2359, which is none of the listed choices.
5	Fractional parts and estimation ｜小數部分與估值	The correct choice is **E**.
6	Triangle side relation ｜三角形邊角關係	$\boxed{B = 2A}$, so the correct choice is **B**.
7	Telescoping product ｜連乘積消去	$\frac{1010}{2019}$.
8	Age ratios ｜年齡比例	As written, $\boxed{S = 49m + 37 \ (m = 0, 1, 2, \ldots)}$. The least possible value is 37.
9	Polynomial coefficients ｜多項式係數	3.
10	Area and trigonometry ｜面積與三角比	$\frac{\sqrt{3}}{8}$.
11	Floor-sum pairing ｜取整和配對	76304.
12	Tiling invariant ｜鋪砌不變量	No, such a tiling is impossible.
13	Rationality identity ｜有理性恒等式	Yes. In fact $S = \frac{\lvert xy + yz + zx\rvert}{\lvert xyz\rvert} \in \mathbb{Q}$.

2019 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Recurring decimals ｜循環小數	$0.0\overline{109}$, so the correct choice is **B**.
2	Egyptian fractions ｜埃及分數	$N = 10$, which is none of the listed choices.

Q	Topic 題型	Answer 答案
3	Reduced fractions and Euler's function ｜既約分數與歐拉函數	672, which is none of the listed choices.
4	Handshake counting ｜握手計數	$S = \dfrac{3n(n-1)}{2}$, which is none of the listed choices.
5	Ellipse and focal property ｜橢圓與焦點性質	$\boxed{10 - 2\sqrt{10}}$.
6	Polynomial functional equation ｜多項式函數方程	$\boxed{P(x) \equiv 2 + \sqrt{5} \text{ or } P(x) \equiv 2 - \sqrt{5}}$.
7	Maximum under a cubic constraint ｜三次約束下的最大值	$\boxed{2}$.
8	Exponential Diophantine equation ｜指數丟番圖方程	$\boxed{(x, y) = (1, 1), (16, 2)}$.
9	Extremal residues modulo 3 ｜模 3 極值問題	$\boxed{n = 6}$.
10	Nonlinear recurrence and hidden linearity ｜非線性遞推與隱藏線性	$\boxed{S_n = 0}$ for every $n \geq 1$.
11	Quadratic conjugates and recurrence ｜二次共軛與遞推	The units digit is $\boxed{2}$.
12	Perfect squares and powers of two ｜完全平方與二次冪	$\boxed{n = 11}$.
13	A-excircle inequality ｜ A-旁切圓不等式	The stated inequality holds, with equality when $B = C$.
14	Unit-distance incidence bound ｜單位距離關聯上界	$\boxed{1974}$.
15	Equal-sum partition ｜等和分組	Yes. The table above gives an explicit partition into eleven groups, each of sum 671.

2020 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Addition chains ｜加法鏈	The sequence displayed above is a valid construction.
2	Consecutive squares ｜連續整數平方	$2, 3, 4, 5, 6$, with $3^2 + 6^2 = 2^2 + 4^2 + 5^2$.
3	Cubic residues ｜立方剩餘	There are no integer solutions.
4	Rational inequality ｜有理不等式	The inequality holds; equality occurs when $a = b$ or $ab = 1$.
5	Magic-sum tables ｜定和方格	$N = 21$.
6	Angle chasing and the sine rule ｜角度追蹤與正弦定理	$AC \perp BD$.
7	Repeated blocks and divisibility ｜循環數字與整除	Divisibility by 17 need not hold, so the correct choice is **D**.
8	Exponential prime equation ｜指數型質數方程	9, which corresponds to **E** (none of the listed numerical choices).

Q	Topic 題型	Answer 答案
9	Symmetric power sums ｜對稱冪和	$N = 4$, so the correct choice is **B**.
10	Symmetric identities ｜對稱恆等式	8, so the correct choice is **B**.
11	Barycentric ratios ｜重心坐標比值	1, so the correct choice is **D**.
12	Prime powers and squares ｜質數冪與平方數	As printed, $p + m \in \{6, 7\}$. Under the likely intended condition $m > 1$, $p + m = 7$.
13	Symmetric rational equations ｜對稱分式方程	$(a, b) = \left(\frac{5+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$ or the reversed pair.
14	Sums of three primes ｜三質數和	As printed, no such two-digit number exists. If "distinct primes" was intended, the answer is 11.
15	Cubic factorisation ｜三次式因式分解	$\boxed{-3abxy(a-b)(x-y)}$.
16	Real algebraic system ｜實數代數方程組	$(a, b) = (-5, 3)$ or $(5, -3)$.
17	Sum of two squares ｜兩平方和	For example, $(x, y) = (16, 42)$.
18	Affine geometry and line ratios ｜仿射幾何與截線比	$BD : DE : EF = 11 : 16 : 6$.
19	Integer roots and Vieta ｜整數根與韋達定理	$S = \frac{32}{5}$.

2020 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Derangements ｜錯排	44 operations.
2	Nonlinear recurrence and induction ｜非線性遞推與歸納	The inequality holds for every $k \geq 0$.
3	Cauchy inequality ｜柯西不等式	The inequality is proved.
4	Cyclic geometry and incircles ｜圓內接四邊形與內切圓	An external common tangent of the two incircles is parallel to BD.
5	Cubic identities and integer equations ｜立方恆等式與整數方程	(a) $3, 4, 5, 6$. (b) $(t, 1-t)$ for $t \in \mathbb{Z}$, and $(-1, -1)$.
6	Alternating square sum ｜平方差交錯和	$N = 2,041,210$, so the correct choice is **E**.
7	Partial fractions ｜部分分式	16, so the correct choice is **D**.
8	Cyclic rational identity ｜循環分式恆等式	1, so the correct choice is **C**.
9	Four-cube identity ｜四立方恆等式	$24abc$, so the correct choice is **D**.
10	Polynomial optimisation ｜多項式最值	-3.
11	Terminal digits ｜末位數字	$a_{2020} + a_{2021} = 6$.

Q	Topic 題型	Answer 答案
12	Cubic factorisation 丨三次因式分解	(a) 0. (b) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.
13	Five cubes representation 丨五個立方數表示	$2020 = 10^3 + 10^3 + 3^3 + 1^3 + (-2)^3$.
14	Real algebraic system 丨實數代數方程組	$(a, b) = (-5, 3)$ or $(5, -3)$.
15	Minimax approximation 丨極小極大逼近	$\min M(b, c) = 1$.
16	Interchanging infinite sums 丨交換無窮和	$S = \dfrac{2019}{2020}$.

2021 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Integer systems 丨整數方程組	5, so the correct choice is **E**.
2	Counting triangles 丨三角形計數	27, so the correct choice is **C**.
3	Factor comparison 丨因式比較	$a > b$, so the correct choice is **A**.
4	Aliquot sums 丨真因數和	2, so the correct choice is **B**.
5	Areas and midpoints 丨面積與中點	$S = 32$.
6	Telescoping product 丨望遠鏡乘積	$P = \dfrac{1011}{2021}$.
7	Linear equations 丨一次方程組	42.
8	Directed angles 丨有向角	$x = 41°$.
9	Optimising a right triangle 丨直角三角形最值	$\min S = 12$.
10	Extremal selection 丨極值選取	$n = 11$.
11	Transforming roots 丨根的變換	$\boxed{3x^2 - 4x - 1 = 0}$.
12	AM–GM inequality 丨算術幾何平均不等式	The inequality is proved, with equality exactly when $a = b = c$.
13	Ptolemy inequality and area 丨托勒密不等式與面積	The required inequality is proved.

2021 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Finite differences and factorials 丨有限差分與階乘	$-\dfrac{2022}{2021!}$.

Q	Topic 題型	Answer 答案
2	The range of quadratic roots ｜二次方程根的範圍	$\boxed{[-1-\sqrt{3},\ 1+\sqrt{3}]}$.
3	Distance from lattice points to a line ｜格點到直線的距離	$\min f(a,b) = \frac{1}{5}$.
4	Triangles in a triangular lattice ｜三角形網格計數	$N_n = \left\lfloor \frac{n(n+2)(2n+1)}{8} \right\rfloor$.
5	Cyclic and homogeneous inequalities ｜循環與齊次不等式	Both inequalities hold; equality in each occurs when $x = y = z$ or $a = b = c$, respectively.
6	Functional pairing ｜函數配對	1010.
7	Average divisor counts ｜平均因數個數	Both strict increases and strict decreases occur infinitely often.
8	A hidden angle bisector ｜隱藏角平分線	DH bisects $\angle EDF$.
9	A polynomial difference operator ｜多項式差分算子	$\boxed{P(x) = A(x^2+x)+C}$ for arbitrary real constants A, C.
10	Tangents, areas, and a right angle ｜切線、面積與直角	$\boxed{\angle BCA = 90^\circ}$.

2022 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Telescoping radicals ｜根式望遠鏡和	$\sqrt{2022}-1$, so the correct choice is **A**.
2	Substitution ｜代換	3, so the correct choice is **C**.
3	Symmetric expressions ｜對稱式	-4044, so the correct choice is **A**.
4	Partial fractions ｜部分分式	3, so the correct choice is **C**.
5	Solution sets of inequalities ｜不等式解集	0, so the correct choice is **A**.
6	Integer-valued rational expressions ｜整值分式	10, so the correct choice is **D**.
7	Fractional parts ｜小數部分	810.
8	Factorisation ｜因式分解	$(x-2y)(x+y-2)$.
9	Sign of a rational expression ｜分式符號	The expression is strictly negative.
10	Parity counting ｜奇偶分類計數	51.
11	Area ratios in a parallelogram ｜平行四邊形面積比	$\frac{1}{6}$.
12	Angle trisectors, median and altitude ｜三等分線、中線與高	$\angle C = 60^\circ$.
13	A factor of a Mersenne-type number ｜梅森型數的因數	One possible divisor is $d = 63$.

Q	Topic 題型	Answer 答案
14	Intersections determined by five points 丨五點連線交點	The number is **not uniquely determined** by the stated conditions. If convex position was intended, the answer is 5.
15	Integer optimisation with congruences 丨同餘限制下的整數最值	366, attained at $(x, y) = (48, 6)$.
16	Maximum area with fixed circumradius 丨定外接圓半徑的最大面積	$3\sqrt{3}$.

2022 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Maximum area with fixed circumradius 丨定外接圓半徑的最大面積	$3\sqrt{3}$.
2	Parity counting 丨奇偶分類計數	51.
3	Minimum intercept area 丨截距三角形的最小面積	24.
4	A cubic minimum 丨三次式最小值	-3, attained at $x = 1$.
5	Induction for the sum of cubes 丨立方和歸納證明	$S_n = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.
6	Intersections determined by five points 丨五點連線交點	The question is underdetermined as printed. If convex position was intended, the answer is 5.
7	A deletion game and pairing strategy 丨刪數遊戲與配對策略	The first player can guarantee a score of 55.
8	A circular letter process 丨圓周字母迭代	No, it is impossible.
9	A cyclic product inequality 丨循環乘積不等式	Both inequalities hold; equality occurs when all relevant adjacent variables are equal.
10	Best constant in an inequality 丨不等式最佳常數	$C = n - 1$.
11	A three-variable product inequality 丨三元乘積不等式	The stated inequality holds, with equality at $x = y = z = 1$.
12	Pairwise coprime denominators 丨兩兩互素分母	The possible positive integers are $\boxed{6, 7, 8}$.

2022/2023 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Divisor counting 丨約數個數	6, so the correct choice is **C**.
2	Signs of cyclic ratios 丨循環比值的符號	2, so the correct choice is **C**.

Q	Topic 題型	Answer 答案
3	Comparing large powers 丨大冪比較	$r < p < q$, so the correct choice is **D**.
4	Nearest perfect cube 丨最接近的完全立方數	2197, so the correct choice is **D**.
5	Symmetric expressions 丨對稱式	723.
6	Diophantine factorisation 丨丟番圖因式分解	2 triples: $(1, 3, 8)$ and $(1, 4, 5)$.
7	Geometric interpretation of radicals 丨根式的幾何詮釋	$\sqrt{65}$.
8	Angle bisector theorem 丨角平分線定理	$AP = 2\sqrt{3} - 3$.
9	Equal-area quadrilateral lemma 丨四邊形等面積引理	Either M is the midpoint of AC or M is the midpoint of BD.
10	Prime arithmetic progressions 丨質數等差數列	$30 \mid d$.
11	Double counting divisor sums 丨約數和的雙重計數	$\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k \left\lfloor \frac{n}{k} \right\rfloor$.
12	Discriminant controlled by a cubic 丨由三次條件控制判別式	0 real roots.
13	Median minimises absolute deviations 丨中位數與絕對值和	2500, attained for every $x \in [50, 51]$.
14	Cyclic divisibility classification 丨循環整除分類	All cyclic rotations of the eleven representatives above; equivalently, 41 ordered quadruples.
15	Symmetric rational identities 丨對稱有理式	$\frac{27}{2}$.
16	Lucas numbers and the golden ratio 丨盧卡斯數與黃金比	The expression equals L_{2n-1}^2, hence is a perfect square.

2022/2023 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Exact trigonometric values 丨特殊三角函數值	$2 - \sqrt{3}$.
2	Change of logarithmic base 丨對數換底	2.
3	Tangent addition identity 丨正切和角恆等式	1.
4	Euler totient double counting 丨歐拉函數雙重計數	$\frac{n(n+1)}{2}$.
5	A product-constrained rational inequality 丨乘積限制有理不等式	1, attained when $a = b = c$.

Q	Topic 題型	Answer 答案
6	Diagonal inequalities in a quadrilateral 丨四邊形對角線不等式	$\max(AB+CD, AD+BC) < AC+BD < AB+BC+CD+DA$.
7	Sharp constant inequality 丨最佳常數不等式	$k=\frac{27}{4}$.
8	Cubic norm identity 丨三次範數恆等式	S is closed under multiplication.
9	Additive decomposition of an array 丨矩陣的可加分解	One valid choice is $x_i = a_{i1} - a_{11}$ and $y_j = a_{1j}$.
10	A central Delannoy identity 丨中央 Delannoy 恆等式	The two sums are equal for every positive integer n.
11	Classifying a recursive sequence 丨遞推數列分類	$a_n = n$ for all $n \geq 1$.
12	Orthocenter vector identity 丨垂心向量恆等式	The zero vector 0.

2023/2024 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Divisor counting 丨約數個數	16, so the correct choice is **D**.
2	A chain of inequalities 丨連鎖不等式	There is exactly 1 real number, so the correct choice is **B**.
3	Right angles in a convex octagon 丨凸八邊形的直角	3, so the correct choice is **C**.
4	Four-digit numbers with digit sum eight 丨數字和為八的四位數	120, so the correct choice is **C**.
5	A sum of squares equal to zero 丨平方和為零	$\frac{5}{9}$.
6	Equal selling prices with opposite percentage changes 丨同售價的盈虧問題	A net loss of \$180.
7	One root is the square of the other 丨一根為另一根的平方	15.
8	Coloured balls and a linear system 丨彩球與線性方程組	70 balls.
9	Area of a right trapezium 丨直角梯形面積	684.
10	Squares on an equilateral triangle 丨等邊三角形三邊上的正方形	$3\sqrt{4+\sqrt{3}}$.
11	A cyclic cubic system 丨循環三次方程組	$(a,b,c)=(0,0,0)$.
12	Mutual divisibility conditions 丨互相整除條件	$(a,b)=(3,2),(10,9),(13,6)$.

Q	Topic 題型	Answer 答案
13	Angle bisector and perpendicular bisector 丨角平分線與垂直平分線	$\frac{ED}{DA} = \frac{BC}{AC - BC}$ for the configuration shown.
14	A locus defined by a $45°$ angle 丨四十五度圓周角軌跡	$\sqrt{5} - \sqrt{2}$.
15	A two-variable rational inequality 丨二元分式不等式	The minimum value is 8, attained at $x = y = 2$.
16	Maximising a sum of reciprocals 丨單位分數和的最大值	$\max S(a, b, c) = \frac{41}{42}$, attained at $(7, 3, 2)$.

2023/2024 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Iteration of a rational radical function 丨根式函數的迭代	$f_n(x) = \frac{x}{\sqrt{1 + nx^2}}$, so the correct choice is **D**.
2	Logarithms with different bases 丨不同底數的對數	1125, so the correct choice is **B**.
3	A product recurrence and a weighted geometric sum 丨乘積遞推與加權等比和	$\frac{2025}{2023}$, so the correct choice is **C**.
4	Repeated differences and the pigeonhole principle 丨重複差值與鴿巢原理	There exists an integer n such that $d_i = n$ for at least 15 indices.
5	Maximum of a sum of square roots 丨根式和的最大值	$k_{\max} = \sqrt{6}$.
6	A hidden perfect square 丨隱藏的完全平方數	$a + b + c = (a - b)^2$.
7	Quadratic residues modulo an odd prime 丨奇素數模下的二次剩餘	Exactly one of $ac + bd$ and $ad + bc$ is a multiple of p.
8	Divisibility of every consecutive block 丨所有連續區段的整除性	No such sequence exists.
9	A cyclic system of four positive variables 丨四個正變量的循環方程組	$(x, y, z, w) = (12, 12, 12, 12)$, or $(12 + \sqrt{143}, 12 - \sqrt{143}, 12 + \sqrt{143}, 12 - \sqrt{143})$, or the tuple obtained by interchanging the two alternating values.
10	Incenter, median and circumcircle 丨內心、中線與外接圓	$\frac{ED}{EI} = \frac{IC}{IB}$.
11	A nonlinear recurrence inequality 丨非線性數列不等式	$a_{2023} \le 1$.

2024/2025 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Telescoping radicals 丨望遠鏡根式和	11, so the correct choice is **D**.
2	Double counting 丨重複計數	64 students; therefore the correct choice is **E** (none of the listed numbers).
3	Symmetric expressions 丨對稱式	-52.
4	Reciprocal substitution 丨倒數代換	1.
5	Biquadratic equations 丨雙二次方程	$x = \pm\frac{1}{2}$.
6	Sums of positive cubes 丨正整數立方和	$n = 4$.
7	Nested radicals 丨嵌套根式	0.
8	Comparing large powers 丨大冪比較	3^{21} is larger.
9	Integer factorisation 丨整數因式分解	180.
10	Digit sums 丨數字和	$n = 1998$ or $n = 2016$.
11	Tangential polygons 丨切圓多邊形	$\boxed{16(\sqrt{2}-1)}$.
12	Cubic differences and integer bounds 丨立方差與整數估計	$(a, b) = (5, 2)$.
13	Angle chasing in an equilateral triangle 丨等邊三角形角度追蹤	$30°$.
14	Vieta's formulas 丨韋達定理	3.
15	Cyclic fractions and divisibility 丨循環分式與整除性	$a + b + c + d$ is composite.
16	Elementary optimisation and inequalities 丨初等最值與不等式	$\max\limits_{0\le x\le 1}(x - x^4) = \frac{3\sqrt[3]{2}}{8}$, and the stated inequality follows.

2024/2025 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Telescoping radicals 丨望遠鏡根式和	22, so the correct choice is **A**.
2	Radicals with negative variables 丨含負變量的根式	$-9x\sqrt{-x}$, so the correct choice is **B**.
3	Logarithms and inequalities 丨對數與不等式	$\max\big(\log_{10} a \log_{10} b\big) = \log_{10} 4$.
4	Cubic equations 丨三次方程	$x = \frac{1}{2}$.
5	Optimisation and inequality 丨最值與不等式	$\max\limits_{[0,1]}(x - x^4) = \frac{3\sqrt[3]{2}}{8}$, which yields the required inequality.
6	Coordinate geometry and reflection 丨坐標幾何與反射	$\angle BAM = \angle CAN$.

Q	Topic 題型	Answer 答案
7	Permutations and the triangle inequality 丨排列與三角不等式	$\min S = 2(n-1) = 2n-2$.
8	Complex geometry 丨複數幾何	PY passes through the midpoint of EF.
9	Factorials and modular arithmetic 丨階乘與模算術	$(a,b,c) = (5,8,14)$.
10	Homogenisation and Schur's inequality 丨齊次化與 Schur 不等式	$a^4+b^4+c^4 \geq a+b+c$.

2025/2026 - Junior Division 初中組

Q	Topic 題型	Answer 答案
1	Geometric series 丨等比級數	$N = 64$, so the correct choice is **D**.
2	Floor functions and radicals 丨取整函數與根式	$N = 2$, so the correct choice is **A**.
3	Estimation and irrationality 丨估值與無理數	The false statement is **B**.
4	Divisibility and averaging 丨整除與平均值	$N = \frac{3306}{7}$, which is none of the listed choices.
5	Real algebraic systems 丨實代數方程組	The solutions are exactly the pairs listed above, with $R = \sqrt{a^2+b^2}$.
6	Rational parametrisation 丨有理參數表示	One parametrisation is $(a,b) = \left(\frac{t^2-2t-1}{t^2+1}, \frac{1-2t-t^2}{t^2+1}\right)$ for $t \in \mathbb{Q}$.
7	Modular arithmetic 丨模算術	$N = 63+65 = 128$.
8	A family of quadratics 丨二次函數族	The fixed point is $(2,9)$, and the larger root satisfies $\boxed{-1 < r < 2 \text{ or } r > 5}$.
9	Binary representation and Vieta's formula 丨二進表示與韋達定理	2 ordered divisions.
10	Coordinate geometry on a circle 丨圓上的坐標幾何	$OI \parallel BE$.

2025/2026 - Senior Division 高中組

Q	Topic 題型	Answer 答案
1	Floor functions and radicals 丨取整函數與根式	2, so the correct choice is **A**.
2	Integer equations 丨整數方程	$N = 2$, so the correct choice is **C**.
3	Combinatorics and recurrence 丨組合與遞推	21, which is none of the listed choices.
4	Quadratic polynomials 丨二次多項式	$p(x) = x^2 - 90x - 1$, so the correct choice is **D**.
5	Finite sets and translations 丨有限集合與平移	$17n = 31m$.

Q	Topic 題型	Answer 答案
6	Mathematical games and sums of squares丨數學遊戲與平方和	No. Amy cannot prevent Bob from winning.
7	Parity and prefix sums丨奇偶性與前綴和	The number of good coins is always **odd**.
8	Symmetric inequalities 丨對稱不等式	$\max S = \dfrac{3}{4}$, attained at $x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$.
9	Euclidean and coordinate geometry丨歐氏與坐標幾何	I, M, O are collinear.

Olympiad Knowledge Map | 奧數知識地圖

The map below groups questions by broad Olympiad domain. A question may appear in more than one domain when its solution genuinely combines methods. Page references are live links in the PDF.
下列索引按主要奧數範疇整理題目；若一題同時運用多種方法，可能會在多個範疇重複出現。PDF 中的題號及頁碼均可按下跳轉。

Algebra and Equations 代數與方程 (103 questions)

2019 Junior Division 初中組	Q7 (p. 9), Q8 (p. 10), Q9 (p. 11), Q13 (p. 14).
2019 Senior Division 高中組	Q6 (p. 21), Q8 (p. 23), Q11 (p. 26), Q12 (p. 26).
2020 Junior Division 初中組	Q2 (p. 33), Q4 (p. 35), Q7 (p. 38), Q8 (p. 38), Q9 (p. 39), Q10 (p. 40), Q13 (p. 42), Q15 (p. 44), Q16 (p. 45), Q19 (p. 48).
2020 Senior Division 高中組	Q5 (p. 55), Q7 (p. 56), Q8 (p. 57), Q9 (p. 58), Q10 (p. 58), Q12 (p. 60), Q14 (p. 61).
2021 Junior Division 初中組	Q1 (p. 65), Q3 (p. 66), Q6 (p. 70), Q7 (p. 70), Q9 (p. 72), Q11 (p. 74).
2021 Senior Division 高中組	Q1 (p. 79), Q2 (p. 79), Q3 (p. 80), Q6 (p. 84), Q9 (p. 87).
2022 Junior Division 初中組	Q1 (p. 92), Q2 (p. 92), Q3 (p. 93), Q4 (p. 93), Q5 (p. 94), Q6 (p. 95), Q8 (p. 96), Q9 (p. 97), Q13 (p. 101).
2022 Senior Division 高中組	Q4 (p. 109), Q5 (p. 110), Q9 (p. 114), Q11 (p. 116).
2022/2023 Junior Division 初中組	Q1 (p. 120), Q2 (p. 120), Q3 (p. 121), Q5 (p. 122), Q6 (p. 123), Q12 (p. 128), Q13 (p. 129), Q15 (p. 132), Q16 (p. 133).
2022/2023 Senior Division 高中組	Q2 (p. 135), Q3 (p. 136), Q5 (p. 137), Q8 (p. 140), Q9 (p. 141), Q10 (p. 142), Q11 (p. 143), Q12 (p. 144).
2023/2024 Junior Division 初中組	Q1 (p. 148), Q2 (p. 148), Q5 (p. 150), Q7 (p. 152), Q8 (p. 153), Q11 (p. 156), Q12 (p. 157).
2023/2024 Senior Division 高中組	Q1 (p. 164), Q2 (p. 164), Q6 (p. 167), Q9 (p. 170).
2024/2025 Junior Division 初中組	Q1 (p. 176), Q3 (p. 177), Q4 (p. 178), Q5 (p. 178), Q7 (p. 180), Q9 (p. 182), Q12 (p. 184), Q14 (p. 187), Q15 (p. 188), Q16 (p. 189).
2024/2025 Senior Division 高中組	Q1 (p. 192), Q2 (p. 192), Q3 (p. 193), Q4 (p. 194), Q5 (p. 194), Q9 (p. 202).
2025/2026 Junior Division 初中組	Q1 (p. 206), Q2 (p. 206), Q3 (p. 207), Q5 (p. 209), Q6 (p. 210), Q8 (p. 212), Q9 (p. 213).
2025/2026 Senior Division 高中組	Q1 (p. 219), Q2 (p. 219), Q4 (p. 221).

Number Theory 數論 (64 questions)

2019 Junior Division 初中組	Q1 (p. 5), Q3 (p. 6), Q4 (p. 7), Q8 (p. 10), Q13 (p. 14).
2019 Senior Division 高中組	Q1 (p. 17), Q2 (p. 17), Q3 (p. 18), Q8 (p. 23), Q9 (p. 24), Q11 (p. 26), Q12 (p. 26), Q15 (p. 30).
2020 Junior Division 初中組	Q3 (p. 34), Q7 (p. 38), Q8 (p. 38), Q12 (p. 42), Q14 (p. 43), Q17 (p. 46), Q19 (p. 48).

2020 Senior Division 高中組	Q5 (p. 55), Q11 (p. 59), Q13 (p. 60).
2021 Junior Division 初中組	Q4 (p. 67).
2021 Senior Division 高中組	Q1 (p. 79), Q3 (p. 80), Q4 (p. 81), Q7 (p. 84).
2022 Junior Division 初中組	Q6 (p. 95), Q7 (p. 95), Q13 (p. 101), Q15 (p. 102).
2022 Senior Division 高中組	Q12 (p. 117).
2022/2023 Junior Division 初中組	Q1 (p. 120), Q4 (p. 122), Q6 (p. 123), Q10 (p. 127), Q11 (p. 128), Q14 (p. 130), Q16 (p. 133).
2022/2023 Senior Division 高中組	Q4 (p. 136), Q11 (p. 143).
2023/2024 Junior Division 初中組	Q1 (p. 148), Q4 (p. 150), Q8 (p. 153), Q12 (p. 157), Q16 (p. 161).
2023/2024 Senior Division 高中組	Q6 (p. 167), Q7 (p. 168), Q8 (p. 169).
2024/2025 Junior Division 初中組	Q6 (p. 179), Q8 (p. 181), Q9 (p. 182), Q10 (p. 182), Q12 (p. 184), Q15 (p. 188).
2024/2025 Senior Division 高中組	Q9 (p. 202).
2025/2026 Junior Division 初中組	Q3 (p. 207), Q4 (p. 208), Q7 (p. 211), Q9 (p. 213).
2025/2026 Senior Division 高中組	Q2 (p. 219), Q5 (p. 222), Q6 (p. 223).

Geometry and Trigonometry 幾何與三角 (48 questions)

2019 Junior Division 初中組	Q6 (p. 8), Q10 (p. 12), Q12 (p. 13).
2019 Senior Division 高中組	Q5 (p. 19), Q13 (p. 27), Q14 (p. 28).
2020 Junior Division 初中組	Q6 (p. 36), Q11 (p. 40), Q18 (p. 46).
2020 Senior Division 高中組	Q4 (p. 53).
2021 Junior Division 初中組	Q2 (p. 65), Q5 (p. 68), Q8 (p. 71), Q9 (p. 72), Q13 (p. 76).
2021 Senior Division 高中組	Q3 (p. 80), Q4 (p. 81), Q8 (p. 86), Q10 (p. 88).
2022 Junior Division 初中組	Q11 (p. 98), Q12 (p. 100), Q14 (p. 102), Q16 (p. 104).
2022 Senior Division 高中組	Q1 (p. 107), Q3 (p. 108), Q6 (p. 111).
2022/2023 Junior Division 初中組	Q7 (p. 124), Q8 (p. 124), Q9 (p. 125), Q13 (p. 129).
2022/2023 Senior Division 高中組	Q1 (p. 135), Q3 (p. 136), Q6 (p. 138), Q12 (p. 144).
2023/2024 Junior Division 初中組	Q3 (p. 149), Q9 (p. 154), Q10 (p. 155), Q13 (p. 158), Q14 (p. 159).
2023/2024 Senior Division 高中組	Q10 (p. 171).
2024/2025 Junior Division 初中組	Q11 (p. 183), Q13 (p. 186).
2024/2025 Senior Division 高中組	Q6 (p. 196), Q7 (p. 198), Q8 (p. 199).
2025/2026 Junior Division 初中組	Q6 (p. 210), Q10 (p. 215).
2025/2026 Senior Division 高中組	Q9 (p. 227).

Combinatorics and Games 組合與博弈 (36 questions)

2019 Junior Division 初中組	Q12 (p. 13).
2019 Senior Division 高中組	Q2 (p. 17), Q4 (p. 19), Q9 (p. 24), Q10 (p. 25), Q14 (p. 28), Q15 (p. 30).
2020 Junior Division 初中組	Q1 (p. 33), Q5 (p. 35).
2020 Senior Division 高中組	Q1 (p. 51).
2021 Junior Division 初中組	Q2 (p. 65), Q10 (p. 73).
2021 Senior Division 高中組	Q4 (p. 81), Q7 (p. 84).
2022 Junior Division 初中組	Q10 (p. 98), Q14 (p. 102).
2022 Senior Division 高中組	Q2 (p. 108), Q6 (p. 111), Q7 (p. 112), Q8 (p. 113).
2022/2023 Junior Division 初中組	Q1 (p. 120), Q11 (p. 128).
2022/2023 Senior Division 高中組	Q4 (p. 136), Q9 (p. 141), Q10 (p. 142), Q11 (p. 143).
2023/2024 Junior Division 初中組	Q1 (p. 148), Q4 (p. 150).
2023/2024 Senior Division 高中組	Q4 (p. 166).
2024/2025 Junior Division 初中組	Q2 (p. 176).
2024/2025 Senior Division 高中組	Q7 (p. 198).
2025/2026 Junior Division 初中組	Q9 (p. 213).
2025/2026 Senior Division 高中組	Q3 (p. 220), Q5 (p. 222), Q6 (p. 223), Q7 (p. 224).

Inequalities and Optimisation 不等式與最值 (44 questions)

2019 Junior Division 初中組	Q4 (p. 7).
2019 Senior Division 高中組	Q5 (p. 19), Q7 (p. 22), Q13 (p. 27).
2020 Junior Division 初中組	Q4 (p. 35).
2020 Senior Division 高中組	Q3 (p. 52), Q10 (p. 58), Q15 (p. 62).
2021 Junior Division 初中組	Q9 (p. 72), Q12 (p. 75), Q13 (p. 76).
2021 Senior Division 高中組	Q5 (p. 83), Q7 (p. 84).
2022 Junior Division 初中組	Q5 (p. 94), Q14 (p. 102), Q15 (p. 102), Q16 (p. 104).
2022 Senior Division 高中組	Q1 (p. 107), Q3 (p. 108), Q4 (p. 109), Q6 (p. 111), Q9 (p. 114), Q10 (p. 115), Q11 (p. 116).
2022/2023 Junior Division 初中組	Q14 (p. 130).
2022/2023 Senior Division 高中組	Q5 (p. 137), Q6 (p. 138), Q7 (p. 139).
2023/2024 Junior Division 初中組	Q2 (p. 148), Q5 (p. 150), Q14 (p. 159), Q15 (p. 160), Q16 (p. 161).
2023/2024 Senior Division 高中組	Q4 (p. 166), Q5 (p. 167), Q11 (p. 173).
2024/2025 Junior Division 初中組	Q11 (p. 183), Q12 (p. 184), Q16 (p. 189).
2024/2025 Senior Division 高中組	Q3 (p. 193), Q5 (p. 194), Q7 (p. 198), Q10 (p. 203).
2025/2026 Senior Division 高中組	Q8 (p. 225).

Sequences, Functions and Analysis 數列、函數與分析 (26 questions)

2019 Junior Division 初中組	Q7 (p. 9), Q11 (p. 13).
2019 Senior Division 高中組	Q8 (p. 23), Q10 (p. 25), Q11 (p. 26).
2020 Junior Division 初中組	Q1 (p. 33).
2020 Senior Division 高中組	Q2 (p. 51), Q6 (p. 56), Q10 (p. 58), Q15 (p. 62), Q16 (p. 63).
2021 Junior Division 初中組	Q6 (p. 70).
2021 Senior Division 高中組	Q1 (p. 79), Q4 (p. 81).
2022 Junior Division 初中組	Q1 (p. 92), Q7 (p. 95).
2022 Senior Division 高中組	Q5 (p. 110).
2022/2023 Junior Division 初中組	Q16 (p. 133).
2022/2023 Senior Division 高中組	Q11 (p. 143).
2023/2024 Senior Division 高中組	Q1 (p. 164), Q3 (p. 165), Q11 (p. 173).
2024/2025 Junior Division 初中組	Q1 (p. 176).
2024/2025 Senior Division 高中組	Q1 (p. 192).
2025/2026 Junior Division 初中組	Q1 (p. 206).
2025/2026 Senior Division 高中組	Q3 (p. 220).

Mixed and Foundational Topics 綜合與基礎 (3 questions)

2019 Junior Division 初中組	Q2 (p. 5), Q5 (p. 7).
2023/2024 Junior Division 初中組	Q6 (p. 151).

Editorial Notes Register | 原題編按索引

The following entries identify source-paper statements that are ambiguous, inconsistent, or underdetermined. They are collected here for transparency and quick review.
以下集中列出原試卷中存在歧義、矛盾或條件不足的題目，方便校對及後續修訂。

2019 Junior Division 初中組 - Question 2 (p. 5)

Editorial Note 編按

As printed, the additional phrase saying that all three fractions are already reduced is incompatible with the two stated integer ratios and the sum $\frac{1}{4}$. The intended ratio calculation nevertheless determines the expected answer below.

2019 Junior Division 初中組 - Question 8 (p. 10)

Editorial Note 編按

The printed conditions do not determine a unique value of S. The complete family is given below. If the intended question was to find the least possible integer average exceeding 30, the answer is 37.

2020 Junior Division 初中組 - Question 12 (p. 42)

Editorial Note 編按

As printed, the problem has two solutions: $(p, m) = (5, 1)$ and $(2, 5)$. Hence $p + m$ is not uniquely determined. If the intended additional condition is $m > 1$ (or that m is prime), the expected answer is 7.

2020 Junior Division 初中組 - Question 14 (p. 43)

Editorial Note 編按

The printed question does not say that the three primes must be distinct. With the usual convention that repetitions are allowed, every two-digit integer is a sum of three primes. The likely intended wording is “three distinct primes”, for which the answer is 11.

2021 Junior Division 初中組 - Question 4 (p. 67)

Editorial Note 編按

The English wording says “all positive divisors”, but the printed example $D(6) = 1 + 2 + 3 = 6$ excludes 6 itself. Accordingly, $D(n)$ is interpreted as the sum of the positive proper divisors, consistently with the example.

2022 Junior Division 初中組 - Question 14 (p. 102)

Editorial Note 編按

As printed, the number is not uniquely determined. The condition that no three of the original five points are collinear does not force all five points to be in convex position, nor does it prevent different segment intersections from coinciding. Two valid configurations are shown below.

2022 Senior Division 高中組 - Question 6 (p. 111)

Editorial Note 編按

The printed conditions do not determine a unique answer. A convex set of five points gives 5 crossings, while a valid configuration with one point inside a quadrilateral can give 3 crossings.

2023/2024 Junior Division 初中組 - Question 13 (p. 158)

Editorial Note 編按

The ratio is not a universal numerical constant; it depends on the side lengths of $\triangle ABC$. The data determine the following exact expression. In the diagram E lies beyond B, corresponding to $AC > BC$.

2025/2026 Junior Division 初中組 - Question 6 (p. 210)

Editorial Note 編按

The Chinese statement and the handwritten solution use *rational numbers*. The English line in the printed paper says “integers”, which would make the assertion false.