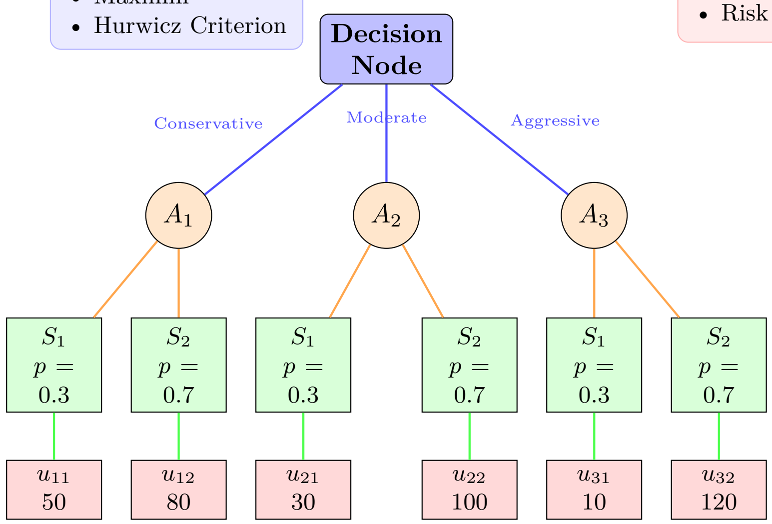
Supplementary of Mathematical Modeling

Decision Theory

Theory and Applications

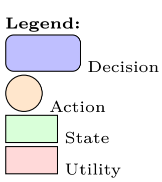
Decision Criteria:
- Expected Utility
- Minimax Regret
- Maximin
- Hurwicz Criterion

Risk Attitudes:
- Risk Averse: $u''(x) < 0$
- Risk Neutral: $u''(x) = 0$
- Risk Seeking: $u''(x) > 0$



Action	State 1	State 2
A_1	50	80
A_2	30	100
A_3	10	120

Expected Utilities:
$\mathbb{E}[U(A_1)] = 0.3(50) + 0.7(80) = 71$
$\mathbb{E}[U(A_2)] = 0.3(30) + 0.7(100) = 79$
$\mathbb{E}[U(A_3)] = 0.3(10) + 0.7(120) = 87$



Expected Utility: $\mathbb{E}[U] = \sum_i p_i \cdot u_i$
Optimal Decision: $a^* = \arg\max_a \mathbb{E}[U(a)]$

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Last update: July 8, 2025

Making Optimal Decisions Under Uncertainty

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Chapter 1

Foundations of Decision Theory

1.1 Introduction to Decision Theory

Decision theory provides a mathematical framework for making optimal choices under uncertainty. It combines probability theory, utility theory, and optimization to help decision-makers select the best course of action when faced with multiple alternatives and uncertain outcomes.

Definition 1.1 (Decision Problem)**.** A decision problem consists of:

- A set of possible actions $A = \{a_1, a_2, \ldots, a_n\}$
- A set of possible states of nature $S = \{s_1, s_2, \ldots, s_m\}$
- A utility function $u : A \times S \to \mathbb{R}$ that assigns a utility value to each action-state pair
- A probability distribution $p(s)$ over the states (if known)

Decision Problem

Investment Portfolio Example: An investor must choose between three investment strategies:

- a_1: Conservative portfolio (60% bonds, 40% stocks)
- a_2: Balanced portfolio (40% bonds, 60% stocks)
- a_3: Aggressive portfolio (20% bonds, 80% stocks)

The economic conditions (states) could be:

- s_1: Economic recession
- s_2: Stable economy
- s_3: Economic boom

1.2 Utility Theory

Definition 1.2 (Utility Function)**.** A utility function $u(x)$ represents the satisfaction or value that a decision-maker derives from outcome x. It captures the decision-maker's preferences and risk attitude.

Theorem

Von Neumann-Morgenstern Expected Utility Theorem: If a decision-maker's preferences satisfy the axioms of completeness, transitivity, continuity, and independence, then there exists a utility function u such that the decision-maker's choices maximize expected utility:

$$\mathbb{E}[U(a)] = \sum_{i=1}^{m} p(s_i) \cdot u(a, s_i)$$

1.2.1 Risk Attitudes and Utility Functions

The curvature of the utility function determines risk attitude:

- **Risk Averse:** $u''(x) < 0$ (concave utility function)
- **Risk Neutral:** $u''(x) = 0$ (linear utility function)
- **Risk Seeking:** $u''(x) > 0$ (convex utility function)

Example 1.1 (Risk Aversion Illustration)**.** Consider two investment options:

- **Option A:** Guaranteed \$50,000
- **Option B:** 50% chance of \$100,000, 50% chance of \$0

Both options have the same expected value: $\mathbb{E}[A] = \mathbb{E}[B] = \$50,000$.
For a risk-averse investor with utility function $u(x) = \sqrt{x}$:

$$u(A) = \sqrt{50,000} = 223.6 \tag{1.1}$$
$$\mathbb{E}[u(B)] = 0.5 \cdot \sqrt{100,000} + 0.5 \cdot \sqrt{0} = 0.5 \cdot 316.2 + 0 = 158.1 \tag{1.2}$$

Since $u(A) > \mathbb{E}[u(B)]$, the risk-averse investor prefers the guaranteed option.

1.2.2 Coefficient of Risk Aversion

Definition 1.3 (Arrow-Pratt Coefficient of Risk Aversion)**.** The coefficient of absolute risk aversion is defined as:

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

The coefficient of relative risk aversion is:

$$r_R(x) = -\frac{xu''(x)}{u'(x)}$$

Example 1.2 (Computing Risk Aversion Coefficients)**.** For the logarithmic utility function $u(x) = \ln(x)$:

$$u'(x) = \frac{1}{x} \tag{1.3}$$
$$u''(x) = -\frac{1}{x^2} \tag{1.4}$$

Therefore:

$$r_A(x) = -\frac{-1/x^2}{1/x} = \frac{1}{x} \tag{1.5}$$

$$r_R(x) = -\frac{x \cdot (-1/x^2)}{1/x} = 1 \tag{1.6}$$

The logarithmic utility exhibits decreasing absolute risk aversion and constant relative risk aversion.

1.3 Decision Criteria Under Uncertainty

1.3.1 Expected Utility Criterion

When probabilities are known, the optimal decision maximizes expected utility:

$$a^* = \text{argmax}_{a \in A} \mathbb{E}[U(a)] = \text{argmax}_{a \in A} \sum_{i=1}^{m} p(s_i) \cdot u(a, s_i)$$

Example 1.3 (Expected Utility Calculation)**.** Consider the investment problem with the following payoff matrix (in thousands of dollars):

	Recession (s_1)	Stable (s_2)	Boom (s_3)
Conservative (a_1)	30	50	60
Balanced (a_2)	10	60	90
Aggressive (a_3)	-20	80	150

Assume probabilities: $p(s_1) = 0.2$, $p(s_2) = 0.5$, $p(s_3) = 0.3$.
For a risk-neutral investor (linear utility $u(x) = x$):

$$\mathbb{E}[U(a_1)] = 0.2(30) + 0.5(50) + 0.3(60) = 49 \tag{1.7}$$

$$\mathbb{E}[U(a_2)] = 0.2(10) + 0.5(60) + 0.3(90) = 59 \tag{1.8}$$

$$\mathbb{E}[U(a_3)] = 0.2(-20) + 0.5(80) + 0.3(150) = 81 \tag{1.9}$$

The optimal choice is a_3 (aggressive portfolio).

1.3.2 Criteria Under Complete Uncertainty

When probabilities are unknown, several criteria can be applied:

Maximin Criterion (Wald Criterion)

Choose the action that maximizes the minimum possible payoff:

$$a^* = \text{argmax}_{a \in A} \min_{s \in S} u(a, s)$$

Maximax Criterion

Choose the action that maximizes the maximum possible payoff:

$$a^* = \text{argmax}_{a \in A} \max_{s \in S} u(a, s)$$

Minimax Regret Criterion (Savage Criterion)

First, compute the regret matrix where regret is the opportunity loss:

$$R(a, s) = \max_{a' \in A} u(a', s) - u(a, s)$$

Then choose the action that minimizes maximum regret:

$$a^* = \text{argmin}_{a \in A} \max_{s \in S} R(a, s)$$

Detailed Solution

Complete Solution for Investment Example:
Using the payoff matrix from the previous example:
Step 1: Maximin Criterion

$$\min_s u(a_1, s) = \min\{30, 50, 60\} = 30 \tag{1.10}$$
$$\min_s u(a_2, s) = \min\{10, 60, 90\} = 10 \tag{1.11}$$
$$\min_s u(a_3, s) = \min\{-20, 80, 150\} = -20 \tag{1.12}$$

Optimal choice: a_1 (conservative), $\max\{30, 10, -20\} = 30$.
Step 2: Maximax Criterion

$$\max_s u(a_1, s) = \max\{30, 50, 60\} = 60 \tag{1.13}$$
$$\max_s u(a_2, s) = \max\{10, 60, 90\} = 90 \tag{1.14}$$
$$\max_s u(a_3, s) = \max\{-20, 80, 150\} = 150 \tag{1.15}$$

Optimal choice: a_3 (aggressive), $\max\{60, 90, 150\} = 150$.
Step 3: Minimax Regret Criterion
First, compute maximum payoffs for each state:

$$\max_a u(a, s_1) = 30 \quad \text{(from } a_1\text{)} \tag{1.16}$$
$$\max_a u(a, s_2) = 80 \quad \text{(from } a_3\text{)} \tag{1.17}$$
$$\max_a u(a, s_3) = 150 \quad \text{(from } a_3\text{)} \tag{1.18}$$

Regret matrix:

	s_1	s_2	s_3
a_1	$30 - 30 = 0$	$80 - 50 = 30$	$150 - 60 = 90$
a_2	$30 - 10 = 20$	$80 - 60 = 20$	$150 - 90 = 60$
a_3	$30 - (-20) = 50$	$80 - 80 = 0$	$150 - 150 = 0$

Maximum regrets: $\max_s R(a_1, s) = 90$, $\max_s R(a_2, s) = 60$, $\max_s R(a_3, s) = 50$.
Optimal choice: a_3 (aggressive), $\min\{90, 60, 50\} = 50$.

Hurwicz Criterion

Combines optimistic and pessimistic approaches with coefficient $\alpha \in [0, 1]$:

$$a^* = \text{argmax}_{a \in A} \left[\alpha \max_{s \in S} u(a, s) + (1 - \alpha) \min_{s \in S} u(a, s) \right]$$

When $\alpha = 1$, this reduces to maximax; when $\alpha = 0$, it reduces to maximin.

Exercise 1.1. For the investment example, compute the optimal decision using the Hurwicz criterion with $\alpha = 0.6$. Interpret the economic meaning of this coefficient.

1.4 The Value of Information

Definition 1.4 (Expected Value of Perfect Information (EVPI))**.** The EVPI represents the maximum amount a decision-maker should be willing to pay for perfect information about the state of nature:

$$\text{EVPI} = \mathbb{E}[U^*] - \max_a \mathbb{E}[U(a)]$$

where $\mathbb{E}[U^*] = \sum_{i=1}^{m} p(s_i) \max_a u(a, s_i)$ is the expected utility with perfect information.

Example 1.4 (Computing EVPI)**.** Using our investment example:

Step 1: Expected utility with perfect information

$$\mathbb{E}[U^*] = p(s_1) \max_a u(a, s_1) + p(s_2) \max_a u(a, s_2) + p(s_3) \max_a u(a, s_3) \tag{1.19}$$

$$= 0.2(30) + 0.5(80) + 0.3(150) \tag{1.20}$$

$$= 6 + 40 + 45 = 91 \tag{1.21}$$

Step 2: Maximum expected utility without information From our earlier calculation: $\max_a \mathbb{E}[U(a)] = 81$ (achieved by a_3).

Step 3: EVPI calculation EVPI $= 91 - 81 = 10$ thousand dollars.

This means the investor should be willing to pay up to \$10,000 for perfect information about future economic conditions.

Chapter 2

Game Theory and Strategic Decisions

2.1 Introduction to Game Theory

Game theory extends decision theory to strategic situations where the outcome depends not only on your decision but also on the decisions of other rational players.

Definition 2.1 (Strategic Form Game)**.** A strategic form game consists of:

- A finite set of players $N = \{1, 2, \ldots, n\}$
- A set of pure strategies S_i for each player $i \in N$
- A payoff function $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for each player i

2.2 Nash Equilibrium

Definition 2.2 (Nash Equilibrium)**.** A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a Nash equilibrium if for every player i:

$$u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_i, \ldots, s_n^*)$$

for all $s_i \in S_i$.

In other words, no player can unilaterally deviate and improve their payoff.

Theorem

Nash's Existence Theorem: Every finite strategic form game has at least one Nash equilibrium (possibly in mixed strategies).

2.2.1 Computing Nash Equilibria in 2×2 Games

Example 2.1 (Finding Pure Strategy Nash Equilibria)**.** Consider the following coordination game:

	Left	Right
Up	(2,1)	(0,0)
Down	(0,0)	(1,2)

Analysis:

- If Player 1 plays Up, Player 2's best response is Left (payoff 1 vs 0)

- If Player 1 plays Down, Player 2's best response is Right (payoff 2 vs 0)
- If Player 2 plays Left, Player 1's best response is Up (payoff 2 vs 0)
- If Player 2 plays Right, Player 1's best response is Down (payoff 1 vs 0)

Nash Equilibria: (Up, Left) and (Down, Right)
Both are coordination equilibria where players coordinate on the same choice.

2.2.2 Mixed Strategy Nash Equilibrium

Definition 2.3 (Mixed Strategy)**.** A mixed strategy for player i is a probability distribution σ_i over the pure strategies in S_i.

Example 2.2 (Computing Mixed Strategy Equilibrium)**.** Consider the Battle of the Sexes game:

	Opera	Football
Opera	(2,1)	(0,0)
Football	(0,0)	(1,2)

Let p = probability that Player 1 plays Opera, and q = probability that Player 2 plays Opera.
Step 1: Player 1's expected payoffs

$$EU_1(\text{Opera}) = 2q + 0(1-q) = 2q \tag{2.1}$$
$$EU_1(\text{Football}) = 0q + 1(1-q) = 1-q \tag{2.2}$$

For mixed strategy equilibrium: $2q = 1 - q \Rightarrow q^* = 1/3$
Step 2: Player 2's expected payoffs

$$EU_2(\text{Opera}) = 1p + 0(1-p) = p \tag{2.3}$$
$$EU_2(\text{Football}) = 0p + 2(1-p) = 2-2p \tag{2.4}$$

For mixed strategy equilibrium: $p = 2 - 2p \Rightarrow p^* = 2/3$
Mixed Strategy Nash Equilibrium: $(p^*, q^*) = (2/3, 1/3)$

2.3 Dominant Strategies

Definition 2.4 (Dominant Strategy)**.**

- Strategy s_i **strictly dominates** s_i' if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all s_{-i}
- Strategy s_i **weakly dominates** s_i' if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all s_{-i}, with strict inequality for some s_{-i}

Example 2.3 (Prisoner's Dilemma with Dominant Strategies)**.**

	Cooperate	Defect
Cooperate	(3,3)	(0,5)
Defect	(5,0)	(1,1)

Analysis for Player 1:

- If Player 2 cooperates: Defect gives 5 > 3 (Cooperate)
- If Player 2 defects: Defect gives 1 > 0 (Cooperate)

Defect strictly dominates Cooperate for both players.

Solution by Iterated Elimination: (Defect, Defect) is the unique Nash equilibrium.

This illustrates the **tragedy of cooperation**: rational individual behavior leads to a socially suboptimal outcome.

2.4 Applications in Economics and Business

2.4.1 Cournot Competition

Real-World Application

Cournot Duopoly Model: Two firms compete by choosing quantities simultaneously.
Setup:

- Market demand: $P(Q) = a - bQ$ where $Q = q_1 + q_2$
- Firm i's cost: $C_i(q_i) = c_i q_i$
- Firm i's profit: $\pi_i(q_1, q_2) = q_i[a - b(q_1 + q_2)] - c_i q_i$

Solution: Firm 1's first-order condition:

$$\frac{\partial \pi_1}{\partial q_1} = a - b(q_1 + q_2) - bq_1 - c_1 = 0$$

This gives the reaction function:

$$q_1^*(q_2) = \frac{a - c_1 - bq_2}{2b}$$

Similarly for Firm 2:

$$q_2^*(q_1) = \frac{a - c_2 - bq_1}{2b}$$

Nash Equilibrium quantities:

$$q_1^* = \frac{a - 2c_1 + c_2}{3b} \tag{2.5}$$

$$q_2^* = \frac{a - 2c_2 + c_1}{3b} \tag{2.6}$$

2.4.2 Auction Theory

Case Study

First-Price Sealed-Bid Auction:
Setting: n bidders compete for a single item. Each bidder i has private valuation v_i drawn from $[0, 1]$ with uniform distribution.
Strategy: Each bidder submits a sealed bid b_i. Highest bidder wins and pays their bid.
Symmetric Nash Equilibrium: In equilibrium, each bidder uses the same bidding function $b(v)$.

The optimal bidding strategy is:

$$b^*(v) = \frac{n-1}{n} \cdot v$$

Key Insights:

- Bidders shade their bids below their true valuation
- More competition (higher n) leads to higher bids
- As $n \to \infty$, $b^*(v) \to v$ (bids approach valuations)

Expected Revenue:

$$R = \mathbb{E}[\max\{b_1, \dots, b_n\}] = \frac{n-1}{n+1}$$

Exercise 2.1. Consider a market with two firms producing differentiated products. The demand functions are:

$$q_1 = a - p_1 + \theta p_2 \tag{2.7}$$

$$q_2 = a - p_2 + \theta p_1 \tag{2.8}$$

where $\theta \in [0, 1]$ measures the degree of substitutability.

Find the Nash equilibrium prices when firms compete simultaneously in prices (Bertrand competition). Analyze how the equilibrium changes as θ varies from 0 (independent products) to 1 (perfect substitutes).

Chapter 3

Multi-Criteria Decision Analysis

3.1 Introduction to MCDA

Multi-Criteria Decision Analysis (MCDA) addresses decisions involving multiple, often conflicting objectives. Unlike single-objective optimization, MCDA seeks solutions representing the best trade-offs among competing criteria.

Definition 3.1 (Multi-Criteria Decision Problem)**.** A multi-criteria decision problem consists of:

- A set of alternatives $A = \{a_1, a_2, \ldots, a_n\}$
- A set of criteria $C = \{c_1, c_2, \ldots, c_m\}$
- A performance matrix X where x_{ij} represents the performance of alternative a_i on criterion c_j
- A set of weights $w = (w_1, w_2, \ldots, w_m)$ with $\sum_{j=1}^{m} w_j = 1$

3.2 Pareto Optimality

Definition 3.2 (Pareto Dominance)**.** Alternative a_i **Pareto dominates** alternative a_j (written $a_i \succ a_j$) if:

- $x_{ik} \geq x_{jk}$ for all criteria k (at least as good in all criteria)
- $x_{ik} > x_{jk}$ for at least one criterion k (strictly better in at least one criterion)

Definition 3.3 (Pareto Optimal Solution)**.** An alternative a_i is **Pareto optimal** (or Pareto efficient) if no other alternative Pareto dominates it.

Example 3.1 (Pareto Frontier Analysis)**.** Consider three investment options evaluated on two criteria:

Alternative	Expected Return (%)	Risk (Std Dev %)
A	8	15
B	12	20
C	10	25
D	6	18

Dominance Analysis:

- A vs B: B has higher return and higher risk (no dominance)
- A vs C: A has lower return but much lower risk (no dominance)
- A vs D: A has higher return and lower risk (A dominates D)
- B vs C: B has higher return and lower risk (B dominates C)
- B vs D: B has higher return but higher risk (no dominance)
- C vs D: C has higher return but higher risk (no dominance)

Pareto Optimal Set: {A, B} (C and D are dominated)

3.3 MCDA Methods

3.3.1 Simple Additive Weighting (SAW)

The weighted sum method combines all criteria into a single score:

$$S(a_i) = \sum_{j=1}^{m} w_j \cdot \bar{x}_{ij}$$

where $\bar{x}_{ij}$ is the normalized performance of alternative i on criterion j.

Example 3.2 (Complete SAW Analysis)**.** Consider laptop selection with the following data:

Laptop	Price ($)	Performance	Battery (hrs)	Weight (kg)
A	800	85	8	2.1
B	1200	95	6	1.8
C	1000	90	10	2.3

Weights: Price (30%), Performance (40%), Battery (20%), Weight (10%)
Step 1: Normalize the matrix
For benefit criteria (Performance, Battery) - higher is better:

$$\bar{x}_{ij} = \frac{x_{ij}}{\max_i x_{ij}}$$

For cost criteria (Price, Weight) - lower is better:

$$\bar{x}_{ij} = \frac{\min_i x_{ij}}{x_{ij}}$$

Normalized matrix:

Laptop	Price	Performance	Battery	Weight
A	1.00	0.89	0.80	0.86
B	0.67	1.00	0.60	1.00
C	0.80	0.95	1.00	0.78

Step 2: Calculate weighted scores

$$S(A) = 0.30(1.00) + 0.40(0.89) + 0.20(0.80) + 0.10(0.86) = 0.902 \tag{3.1}$$

$$S(B) = 0.30(0.67) + 0.40(1.00) + 0.20(0.60) + 0.10(1.00) = 0.821 \tag{3.2}$$

$$S(C) = 0.30(0.80) + 0.40(0.95) + 0.20(1.00) + 0.10(0.78) = 0.898 \tag{3.3}$$

Ranking: A (0.902) $>$ C (0.898) $>$ B (0.821)

3.3.2 TOPSIS Method

TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) ranks alternatives based on their distance from ideal and anti-ideal solutions.

1: Normalize the decision matrix
2: Calculate weighted normalized matrix
3: Determine positive ideal solution (PIS) and negative ideal solution (NIS)
4: Calculate separation measures from PIS and NIS
5: Calculate relative closeness to ideal solution
6: Rank alternatives by closeness index

Detailed Solution

Complete TOPSIS Solution for Laptop Example:
Step 1 & 2: Use the weighted normalized matrix from SAW:

Laptop	Price	Performance	Battery	Weight
A	0.30	0.356	0.16	0.086
B	0.20	0.40	0.12	0.10
C	0.24	0.38	0.20	0.078

Step 3: Determine ideal solutions

$$A^+ = (\max\{0.30, 0.20, 0.24\}, \max\{0.356, 0.40, 0.38\}, \max\{0.16, 0.12, 0.20\}, \max\{0.086, 0.10, 0.078\}) \quad (3.4)$$

$$= (0.30, 0.40, 0.20, 0.10) \quad (3.5)$$

$$A^- = (0.20, 0.356, 0.12, 0.078) \quad (3.6)$$

Step 4: Calculate separation measures

$$d_A^+ = \sqrt{(0.30-0.30)^2 + (0.356-0.40)^2 + (0.16-0.20)^2 + (0.086-0.10)^2} = 0.0616 \quad (3.7)$$

$$d_B^+ = \sqrt{(0.20-0.30)^2 + (0.40-0.40)^2 + (0.12-0.20)^2 + (0.10-0.10)^2} = 0.1281 \quad (3.8)$$

$$d_C^+ = \sqrt{(0.24-0.30)^2 + (0.38-0.40)^2 + (0.20-0.20)^2 + (0.078-0.10)^2} = 0.0721 \quad (3.9)$$

$$d_A^- = \sqrt{(0.30-0.20)^2 + (0.356-0.356)^2 + (0.16-0.12)^2 + (0.086-0.078)^2} = 0.1077 \quad (3.10)$$

$$d_B^- = \sqrt{(0.20-0.20)^2 + (0.40-0.356)^2 + (0.12-0.12)^2 + (0.10-0.078)^2} = 0.0494 \quad (3.11)$$

$$d_C^- = \sqrt{(0.24-0.20)^2 + (0.38-0.356)^2 + (0.20-0.12)^2 + (0.078-0.078)^2} = 0.0894 \quad (3.12)$$

Step 5: Calculate closeness coefficients

$$CC_A = \frac{d_A^-}{d_A^+ + d_A^-} = \frac{0.1077}{0.0616 + 0.1077} = 0.636 \quad (3.13)$$

$$CC_B = \frac{d_B^-}{d_B^+ + d_B^-} = \frac{0.0494}{0.1281 + 0.0494} = 0.278 \quad (3.14)$$

$$CC_C = \frac{d_C^-}{d_C^+ + d_C^-} = \frac{0.0894}{0.0721 + 0.0894} = 0.554 \quad (3.15)$$

TOPSIS Ranking: A (0.636) > C (0.554) > B (0.278)

3.3.3 Analytic Hierarchy Process (AHP)

AHP uses pairwise comparisons to determine criteria weights and alternative preferences.

Definition 3.4 (Pairwise Comparison Matrix)**.** A pairwise comparison matrix M is an $n \times n$ matrix where m_{ij} represents the relative importance of criterion i compared to criterion j:

- $m_{ij} = 1/m_{ji}$ (reciprocal property)
- $m_{ii} = 1$ (diagonal elements)
- $m_{ij} > 0$ for all i, j

Definition 3.5 (Consistency Ratio)**.** The consistency ratio measures the consistency of pairwise comparisons:

$$CR = \frac{CI}{RI}$$

where $CI = \frac{\lambda_{\max} - n}{n-1}$ is the consistency index and RI is the random consistency index.

A $CR < 0.10$ indicates acceptable consistency.

Example 3.3 (AHP Weight Calculation)**.** Consider pairwise comparisons for four criteria: Quality (Q), Cost (C), Delivery (D), Service (S).

Pairwise Comparison Matrix:

	Q	C	D	S
Q	1	3	5	7
C	1/3	1	3	5
D	1/5	1/3	1	3
S	1/7	1/5	1/3	1

Step 1: Calculate column sums $\sum_i m_{i1} = 1 + 1/3 + 1/5 + 1/7 = 1.676$

Step 2: Normalize matrix and calculate weights The priority vector (weights) is obtained as the principal eigenvector: $w = (0.558, 0.264, 0.129, 0.049)$

Step 3: Check consistency $\lambda_{\max} = 4.037$, $CI = 0.012$, $RI = 0.90$, $CR = 0.014 < 0.10$

3.4 Sensitivity Analysis in MCDA

Exploration

Weight Sensitivity Analysis:
Methodology:

1. Vary each weight systematically (e.g., ±10%, ±20%)
2. Recalculate alternative rankings
3. Identify critical weight ranges where ranking changes
4. Determine robustness of the solution

Key Questions:

- Which weights most affect the ranking?
- What is the minimum weight change needed to change the top alternative?
- Are there dominant alternatives that remain optimal across weight variations?

Practical Importance: Real decision-makers often have imprecise preferences. Sensitivity analysis helps identify when these imprecisions matter for the final decision.

Exercise 3.1. A company is selecting a supplier based on four criteria: Quality (40%), Cost (30%), Delivery (20%), and Flexibility (10%).

Supplier	Quality	Cost ($)	Delivery (days)	Flexibility
A	95	1000	5	8
B	90	800	7	9
C	98	1200	3	7

1. Apply both SAW and TOPSIS methods 2. Compare the rankings obtained 3. Perform sensitivity analysis by varying the Quality weight from 20% to 60% 4. Discuss the practical implications of any ranking changes

Chapter 4

Behavioral Decision Theory

4.1 Introduction to Behavioral Decision Theory

Traditional decision theory assumes perfectly rational decision-makers with stable preferences, unlimited cognitive capacity, and complete information processing abilities. However, extensive experimental evidence shows systematic deviations from these assumptions.

Definition 4.1 (Descriptive vs. Normative Theory)**.**

- **Normative Theory:** Prescribes how rational agents *should* make decisions
- **Descriptive Theory:** Describes how people *actually* make decisions
- **Prescriptive Theory:** Provides practical guidance for improving decision-making

4.2 Prospect Theory

Theorem

Prospect Theory (Kahneman & Tversky, 1979):
Prospect Theory describes decision-making under risk through two phases:

1. **Editing Phase:** Restructure the decision problem using operations like coding, combination, segregation, and cancellation
2. **Evaluation Phase:** Evaluate prospects using a value function and probability weighting function

The overall value of a prospect is:

$$V = \sum_i w(p_i)v(x_i)$$

where $w(p_i)$ is the decision weight and $v(x_i)$ is the value of outcome x_i.

4.2.1 Value Function Properties

The prospect theory value function has three key features:

Definition 4.2 (Value Function)**.**

$$v(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \text{ (gains)} \\ -\lambda(-x)^\beta & \text{if } x < 0 \text{ (losses)} \end{cases}$$

where:

- **Reference Dependence:** $v(0) = 0$ (outcomes evaluated relative to reference point)
- **Loss Aversion:** $\lambda > 1$ (losses loom larger than gains)
- **Diminishing Sensitivity:** $\alpha, \beta < 1$ (decreasing marginal value)

Example 4.1 (Loss Aversion Demonstration)**.** Consider two prospects:

- **Prospect A:** Gain \$100 with certainty
- **Prospect B:** 50% chance to gain \$200, 50% chance to gain \$0

Both have the same expected value (\$100), but most people prefer A.
Now consider:

- **Prospect C:** Lose \$100 with certainty
- **Prospect D:** 50% chance to lose \$200, 50% chance to lose \$0

Most people prefer D (the risky option) when facing losses.
Explanation: The value function is concave for gains (risk aversion) and convex for losses (risk seeking).

4.2.2 Probability Weighting

Definition 4.3 (Probability Weighting Function)**.**

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$$

Key properties:

- $w(0) = 0$ and $w(1) = 1$
- Inverse S-shaped: overweighting of small probabilities, underweighting of moderate to high probabilities
- Typical parameter: $\gamma \approx 0.61$

Example 4.2 (Probability Weighting Effects)**. Lottery Behavior:**

- Small probability of large gain (lottery): $w(0.001) \approx 0.032$ vs. objective 0.001
- People overweight the tiny chance of winning, making lotteries attractive

Insurance Behavior:

- Small probability of large loss: $w(0.01) \approx 0.079$ vs. objective 0.01
- People overweight small disaster probabilities, making insurance attractive

This explains why the same person might buy both lottery tickets and insurance.

4.3 Cognitive Biases and Heuristics

4.3.1 Availability Heuristic

Definition 4.4 (Availability Heuristic)**.** People estimate the probability of events based on how easily examples come to mind.

Real-World Application

Medical Diagnosis Example:
Emergency room doctors might overestimate the probability of dramatic but rare conditions they've recently encountered, leading to:

- Over-testing for rare diseases
- Under-diagnosis of common conditions
- Systematic bias in probability assessments

Mitigation Strategies:

- Use base rate information systematically
- Implement decision support systems
- Regular calibration training

4.3.2 Anchoring and Adjustment

Definition 4.5 (Anchoring Bias)**.** People make estimates by starting from an initial value (anchor) and adjusting insufficiently.

Example 4.3 (Anchoring in Negotiations)**. Scenario:** Salary negotiation for a position with market value $80,000.

Case 1: Employer opens with $60,000

- Candidate likely to settle around $70,000
- Anchor pulls final agreement down

Case 2: Candidate opens with $95,000

- Employer likely to counter around $85,000
- Anchor pulls final agreement up

Strategic Implication: The first offer significantly influences the final outcome, even when both parties know the market value.

4.3.3 Framing Effects

Definition 4.6 (Framing Effect)**.** Logically equivalent presentations of the same information lead to different decisions.

Example 4.4 (Asian Disease Problem)**.** Imagine the US is preparing for an unusual Asian disease expected to kill 600 people. Two programs are proposed:

Positive Frame:

- Program A: 200 people will be saved
- Program B: 1/3 probability that 600 people will be saved, 2/3 probability that no people will be saved

Negative Frame:

- Program C: 400 people will die
- Program D: 1/3 probability that nobody will die, 2/3 probability that 600 people will die

Results: Most people choose A over B (risk averse for gains) but D over C (risk seeking for losses), even though A C and B D.

Mathematical Analysis: Let $v(\cdot)$ be the prospect theory value function.

For gains (lives saved):

$$v(200) > \frac{1}{3}v(600) + \frac{2}{3}v(0) = \frac{1}{3}v(600)$$

For losses (lives lost):

$$\frac{1}{3}v(-0) + \frac{2}{3}v(-600) > v(-400)$$

This demonstrates how the same decision problem leads to opposite preferences based on framing.

4.4 Bounded Rationality

Definition 4.7 (Bounded Rationality (Herbert Simon))**.** Human decision-making is limited by:

- **Cognitive limitations:** Working memory, processing speed
- **Information constraints:** Incomplete, costly information
- **Time pressure:** Decisions often made under time constraints

4.4.1 Satisficing vs. Optimizing

Definition 4.8 (Satisficing)**.** Choosing the first alternative that meets an aspiration level rather than searching for the optimal solution.

Case Study

House Hunting Example:
Optimizing Strategy:

- Search all available houses
- Evaluate each on all criteria
- Choose the house with highest utility

- Result: Optimal choice but high search costs

Satisficing Strategy:

- Define minimum acceptable standards (aspiration levels)
- Search sequentially
- Choose first house meeting all standards
- Result: "Good enough" choice with lower search costs

When is satisficing optimal?

- High search costs relative to improvement in outcomes
- Time pressure
- When "good enough" solutions exist
- Complex decision environments

4.5 Behavioral Game Theory

4.5.1 Social Preferences

Traditional game theory assumes players care only about their own payoffs. Behavioral game theory incorporates social preferences.

Definition 4.9 (Types of Social Preferences)**.**

- **Altruism:** Positive concern for others' welfare
- **Inequity Aversion:** Dislike of unequal outcomes
- **Reciprocity:** Tendency to respond to kind/unkind actions similarly
- **Fairness:** Preference for procedurally fair outcomes

4.5.2 Experimental Games

Example 4.5 (Ultimatum Game Detailed Analysis)**. Setup:**

- Player 1 (Proposer) suggests how to split $10
- Player 2 (Responder) accepts or rejects
- If rejected, both get $0

Game Theory Prediction:

- Player 1 offers $0.01 to Player 2
- Player 2 accepts (any positive amount better than $0)
- Outcome: ($9.99, $0.01)

Experimental Results:

- Modal offer: $4-5 (40-50% of total)
- Offers below $2 rejected 50-80% of the time
- Average accepted offer: $3-4

Behavioral Explanations:

- **Proposers:** Anticipate rejection of low offers (strategic fairness)
- **Responders:** Willing to pay to punish unfair behavior
- **Cultural variation:** Cross-cultural studies show variation in "fair" shares

Implications for Economics:

- Wage bargaining: Workers may reject "unfair" wage offers
- Contract design: Need to consider fairness perceptions
- Market outcomes: Competition may not eliminate fairness considerations

Example 4.6 (Public Goods Game)**. Setup:**

- n players each receive endowment e
- Decide how much to contribute to public good: $c_i \in [0, e]$
- Total contribution multiplied by $m > 1$
- Total return divided equally among all players
- Player i's payoff: $(e - c_i) + \frac{m}{n} \sum_{j=1}^{n} c_j$

Nash Equilibrium: $c_i = 0$ for all i (free-riding)
Social Optimum: $c_i = e$ for all i (if $m > 1$)
Experimental Results:

- Initial contributions: 40-60% of endowment
- Decay over time towards Nash prediction
- Communication increases cooperation
- Punishment mechanisms sustain cooperation

4.6 Applications to Management and Policy

Real-World Application

Nudging and Choice Architecture:
Concept: Design decision environments to help people make better choices without eliminating options.

Examples:

1. **Default Options:** Automatic enrollment in 401(k) plans increases participation rates from 20% to 90%
2. **Framing:** "90% fat-free" vs. "10% fat" - same information, different appeal
3. **Social Norms:** "Most guests reuse towels" increases compliance more than environmental appeals
4. **Loss Framing:** "You lose $50/month by not switching" vs. "Save $50/month by switching"

Ethical Considerations:

- Transparency: Should people know they're being nudged?
- Autonomy: Does nudging respect individual choice?
- Effectiveness: Do nudges work long-term?

Exercise 4.1. Design a behavioral intervention for one of the following scenarios:

1. **Energy Conservation:** Increase household energy conservation 2. **Health Behavior:** Encourage healthier food choices in cafeterias 3. **Financial Planning:** Increase retirement savings among young workers

For your chosen scenario:

1. Identify relevant cognitive biases
2. Design specific interventions based on behavioral insights
3. Predict potential challenges and limitations
4. Suggest methods to evaluate effectiveness

Chapter 5

Sequential Decision Making and Dynamic Programming

5.1 Introduction to Sequential Decisions

Many real-world decisions involve sequences of choices over time, where:

- Current decisions affect future options
- Information may be revealed over time
- Future rewards should be discounted
- Learning and adaptation occur

Definition 5.1 (Sequential Decision Problem)**.** A sequential decision problem consists of:

- Time periods $t = 0, 1, 2, \ldots, T$ (finite or infinite horizon)
- State space S describing possible situations
- Action space $A(s)$ available in each state s
- Transition probabilities $P(s'|s, a)$
- Reward function $r(s, a)$ or $r(s, a, s')$
- Discount factor $\delta \in [0, 1]$

5.2 Dynamic Programming

Theorem

Bellman's Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

5.2.1 Finite Horizon Dynamic Programming

For a finite horizon problem with T periods, the value function satisfies:

Definition 5.2 (Bellman Equation - Finite Horizon)**.**

$$V_t(s) = \max_{a \in A(s)} \left[r(s,a) + \delta \sum_{s'} P(s'|s,a) V_{t+1}(s') \right]$$

with boundary condition $V_{T+1}(s) = 0$ for all s.
The optimal policy is:

$$\pi_t^*(s) = \text{argmax}_{a \in A(s)} \left[r(s,a) + \delta \sum_{s'} P(s'|s,a) V_{t+1}(s') \right]$$

Example 5.1 (Inventory Management Problem)**. Problem Setup:**

- State s_t: inventory level at beginning of period t
- Action a_t: order quantity
- Demand d_t is random with known distribution
- Costs: ordering cost $c \cdot a_t$, holding cost $h \cdot \max(s_t + a_t - d_t, 0)$, shortage cost $p \cdot \max(d_t - s_t - a_t, 0)$

Bellman Equation:

$$V_t(s) = \min_{a \geq 0} \left[ca + \mathbb{E}_d[h \max(s + a - d, 0) + p \max(d - s - a, 0)] + \delta \mathbb{E}_d[V_{t+1}(s + a - d)] \right]$$

Optimal Policy Structure: Under certain conditions, the optimal policy has a base-stock form: order up to level S_t^* if current inventory is below s_t^*, otherwise don't order.

5.2.2 Infinite Horizon Dynamic Programming

For infinite horizon problems, the value function satisfies:

Definition 5.3 (Bellman Equation - Infinite Horizon)**.**

$$V(s) = \max_{a \in A(s)} \left[r(s,a) + \delta \sum_{s'} P(s'|s,a) V(s') \right]$$

Under appropriate conditions, there exists a unique solution V^* and a stationary optimal policy π^*.

5.2.3 Value Iteration Algorithm

Initialize $V_0(s)$ for all $s \in S$
while not converged **do**
  for each state $s \in S$ **do**
    $V_{k+1}(s) \leftarrow \max_{a \in A(s)} [r(s,a) + \delta \sum_{s'} P(s'|s,a) V_k(s')]$
  end for
  $k \leftarrow k + 1$

7: **end while**
8: Extract policy: $\pi^*(s) = \text{argmax}_{a \in A(s)} [r(s,a) + \delta \sum_{s'} P(s'|s,a) V^*(s')]$

Example 5.2 (Machine Replacement Problem)**. Setup:**

- Machine ages: $s \in \{1, 2, 3, 4, 5\}$ (5 = broken)
- Actions: Keep ($a = 0$) or Replace ($a = 1$)
- Operating costs: $c(s) = s^2$
- Replacement cost: $R = 10$
- Transition probabilities: Machine ages by 1 year with probability 0.7, breaks with probability 0.3

Value Iteration Solution:
Iteration 1: $V_0(s) = 0$ for all s

$$V_1(1) = \max\{-1 + 0.8 \cdot 0.7 \cdot 0, -10 + 0.8 \cdot 0\} = \max\{-1, -10\} = -1 \quad (5.1)$$
$$V_1(2) = \max\{-4 + 0.8 \cdot 0.7 \cdot 0, -10 + 0.8 \cdot 0\} = \max\{-4, -10\} = -4 \quad (5.2)$$
$$\vdots \quad (5.3)$$

Converged Solution:

- $V^*(1) = -22.5$, $\pi^*(1) = \text{Keep}$
- $V^*(2) = -25.2$, $\pi^*(2) = \text{Keep}$
- $V^*(3) = -27.8$, $\pi^*(3) = \text{Keep}$
- $V^*(4) = -29.1$, $\pi^*(4) = \text{Replace}$
- $V^*(5) = -30.5$, $\pi^*(5) = \text{Replace}$

Optimal Policy: Replace when machine reaches age 4 or breaks.

5.3 Decision Trees

Decision trees provide a graphical method for analyzing sequential decisions under uncertainty.

Definition 5.4 (Decision Tree Components)**.**

- **Decision nodes** (squares): Points where decision-maker chooses
- **Chance nodes** (circles): Points where nature/uncertainty resolves
- **Terminal nodes**: End points with final payoffs
- **Branches**: Represent decisions or chance outcomes

5.3.1 Backward Induction

Start at terminal nodes with known payoffs
Work backward to chance nodes
At each chance node, calculate expected value
Continue backward to decision nodes
At each decision node, choose action with highest expected value
Continue until reaching the root

Example 5.3 (New Product Development Decision)**.** A company considers developing a new product with the following structure:

Stage 1: Decide whether to conduct market research ($50k cost) **Stage 2:** Decide whether to develop product ($200k cost) **Stage 3:** Market outcome (success/failure)

Without Research:

- P(Success) = 0.6, Profit if success = $800k
- P(Failure) = 0.4, Profit if failure = -$100k

With Research:

- P(Positive signal) = 0.7
- If positive signal: P(Success|Positive) = 0.8
- If negative signal: P(Success|Negative) = 0.2

Solution by Backward Induction:

Step 1: Calculate expected values without research

$$EV(\text{Develop}) = 0.6 \times 800 + 0.4 \times (-100) - 200 = 240k$$

$$EV(\text{Don't Develop}) = 0$$

Choose: Develop (EV = $240k)

Step 2: Calculate expected values with research

If positive signal:

$$EV(\text{Develop}|\text{Positive}) = 0.8 \times 800 + 0.2 \times (-100) - 200 = 420k$$

If negative signal:

$$EV(\text{Develop}|\text{Negative}) = 0.2 \times 800 + 0.8 \times (-100) - 200 = -60k$$

Expected value of research strategy:

$$EV(\text{Research}) = 0.7 \times 420 + 0.3 \times 0 - 50 = 244k$$

Optimal Decision: Conduct research ($244k > $240k)

Value of Information: $244k - $240k = $4k

5.4 Markov Decision Processes

Definition 5.5 (Markov Decision Process (MDP))**.** An MDP satisfies the Markov property: the probability of future states depends only on the current state and action, not on the history.

Formally: $P(S_{t+1} = s'|S_t = s, A_t = a, S_{t-1}, A_{t-1}, \ldots) = P(S_{t+1} = s'|S_t = s, A_t = a)$

5.4.1 Policy Types

Definition 5.6 (Policy Classifications)**.**

- **Deterministic vs. Stochastic:** $\pi(s) \in A$ vs. $\pi(a|s) \in [0, 1]$
- **Stationary vs. Non-stationary:** Time-independent vs. time-dependent
- **Markovian vs. History-dependent:** Depends only on current state vs. entire history

Theorem 5.1 (Existence of Optimal Stationary Policy)**.** *For infinite horizon discounted MDPs with finite state and action spaces, there exists an optimal policy that is:*

- *Stationary (time-independent)*
- *Markovian (depends only on current state)*
- *Deterministic (pure strategies)*

Case Study

Optimal Stopping Problem - Job Search:
Setup:

- Job offers arrive with wages w drawn from distribution $F(w)$
- Cost of search per period: c
- Discount factor: δ
- Decision each period: Accept current offer or continue searching

State: Current wage offer w **Actions:** Accept or Reject
Bellman Equation:

$$V(w) = \max\left\{\frac{w}{1-\delta}, -c + \delta \int V(w')dF(w')\right\}$$

Optimal Policy Structure: There exists a reservation wage w^* such that:

- Accept if $w \geq w^*$
- Reject if $w < w^*$

The reservation wage satisfies:

$$\frac{w^*}{1-\delta} = -c + \delta \int V(w')dF(w')$$

Comparative Statics:

- Higher search cost $c \to$ Lower reservation wage
- Higher discount factor $\delta \to$ Higher reservation wage
- Better distribution of offers $\to$ Higher reservation wage

Exercise 5.1. Consider a simplified model of aircraft engine maintenance. An engine can be in one of three states:

- State 1: Good condition
- State 2: Fair condition
- State 3: Poor condition

In each period, you can choose to:

- Continue operating (cost = 0, 5, 15 for states 1, 2, 3)
- Perform maintenance (cost = 10, brings engine to state 1)
- Replace engine (cost = 100, brings engine to state 1)

Transition probabilities for ”continue operating”:

- From state 1: stay in 1 (0.7), move to 2 (0.3)
- From state 2: stay in 2 (0.6), move to 3 (0.4)
- From state 3: engine fails (infinite cost)

Use value iteration with discount factor $\delta = 0.9$ to find the optimal maintenance policy.

5.5 Value of Information

Information has value when it can change decisions or improve outcomes. We distinguish between different types of information value.

Definition 5.7 (Expected Value of Perfect Information (EVPI))**.** The EVPI represents the maximum amount a decision-maker should pay for perfect information:

$$\text{EVPI} = \mathbb{E}[V^*] - V^*_{\text{no info}}$$

where $\mathbb{E}[V^*]$ is the expected value with perfect information and $V^*_{\text{no info}}$ is the optimal value without additional information.

Definition 5.8 (Expected Value of Sample Information (EVSI))**.** The EVSI represents the value of imperfect information from a sample or signal:

$$\text{EVSI} = V^*_{\text{with sample}} - V^*_{\text{no info}}$$

Example 5.4 (Medical Testing Decision)**.** A patient faces a decision about surgery for a condition that occurs in 30% of similar cases.

Without Surgery:

- If condition present: Utility = 0.4
- If condition absent: Utility = 0.9

With Surgery:

- If condition present: Utility = 0.8
- If condition absent: Utility = 0.7

Step 1: Decision without additional information

$$EU(\text{No Surgery}) = 0.3(0.4) + 0.7(0.9) = 0.75 \tag{5.4}$$
$$EU(\text{Surgery}) = 0.3(0.8) + 0.7(0.7) = 0.73 \tag{5.5}$$

Optimal decision: No Surgery (EU = 0.75)

Step 2: Value with perfect information

$$\mathbb{E}[V^*] = 0.3 \cdot \max\{0.4, 0.8\} + 0.7 \cdot \max\{0.9, 0.7\} \tag{5.6}$$
$$= 0.3(0.8) + 0.7(0.9) = 0.87 \tag{5.7}$$

Step 3: EVPI calculation EVPI = 0.87 − 0.75 = 0.12

This means the patient should be willing to pay up to 0.12 utility units for a perfect diagnostic test.

5.5.1 Diagnostic Testing with Imperfect Information

Example 5.5 (Imperfect Test Analysis)**.** Consider a diagnostic test with:

- Sensitivity = 0.9 (probability of positive test given disease)
- Specificity = 0.8 (probability of negative test given no disease)

Step 1: Calculate test result probabilities

$$P(\text{Positive}) = P(+|D)P(D) + P(+|\neg D)P(\neg D) \quad (5.8)$$
$$= 0.9(0.3) + 0.2(0.7) = 0.41 \quad (5.9)$$

Step 2: Update probabilities using Bayes' rule

$$P(D|\text{Positive}) = \frac{P(+|D)P(D)}{P(\text{Positive})} = \frac{0.9 \times 0.3}{0.41} = 0.659 \quad (5.10)$$
$$P(D|\text{Negative}) = \frac{P(-|D)P(D)}{P(\text{Negative})} = \frac{0.1 \times 0.3}{0.59} = 0.051 \quad (5.11)$$

Step 3: Optimal decisions conditional on test results
If test positive:

$$EU(\text{Surgery}|+) = 0.659(0.8) + 0.341(0.7) = 0.766 \quad (5.12)$$
$$EU(\text{No Surgery}|+) = 0.659(0.4) + 0.341(0.9) = 0.571 \quad (5.13)$$

Choose Surgery.
If test negative:

$$EU(\text{Surgery}|-) = 0.051(0.8) + 0.949(0.7) = 0.705 \quad (5.14)$$
$$EU(\text{No Surgery}|-) = 0.051(0.4) + 0.949(0.9) = 0.874 \quad (5.15)$$

Choose No Surgery.
Step 4: Expected value with test

$$V^*_{\text{with test}} = P(+) \cdot EU(\text{Surgery}|+) + P(-) \cdot EU(\text{No Surgery}|-) \quad (5.16)$$
$$= 0.41(0.766) + 0.59(0.874) = 0.830 \quad (5.17)$$

Step 5: EVSI calculation EVSI = 0.830 − 0.75 = 0.08
The imperfect test has value 0.08, which is 67% of the perfect information value.

5.6 Entropy and Information Content

Definition 5.9 (Shannon Entropy)**.** For a discrete random variable X with probability distribution $p(x)$, the entropy is:

$$H(X) = -\sum_x p(x) \log_2 p(x)$$

Entropy measures the uncertainty or information content of a random variable.

Definition 5.10 (Conditional Entropy)**.** The conditional entropy of Y given X is:

$$H(Y|X) = -\sum_{x,y} p(x,y) \log_2 p(y|x)$$

Definition 5.11 (Mutual Information)**.** The mutual information between X and Y measures how much information one variable provides about another:

$$I(X;Y) = H(Y) - H(Y|X) = H(X) - H(X|Y)$$

Example 5.6 (Information Content in Coin Flips)**.** **Fair Coin:** $P(H) = P(T) = 0.5$

$$H = -0.5\log_2(0.5) - 0.5\log_2(0.5) = 1 \text{ bit}$$

Biased Coin: $P(H) = 0.9, P(T) = 0.1$

$$H = -0.9\log_2(0.9) - 0.1\log_2(0.1) = 0.469 \text{ bits}$$

The fair coin has higher entropy (more uncertainty) than the biased coin.

5.7 Learning and Adaptive Decision Making

5.7.1 Bayesian Learning

Definition 5.12 (Bayesian Update)**.** Given prior belief $p(\theta)$ and observation x with likelihood $p(x|\theta)$:

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

where $p(x) = \int p(x|\theta)p(\theta)d\theta$ is the marginal likelihood.

Example 5.7 (Learning About Market Demand)**.** A company launches a new product and wants to learn about demand parameter θ.

Prior: $\theta \sim \text{Beta}(\alpha_0, \beta_0)$ where $\alpha_0 = 2, \beta_0 = 8$ **Observations:** n trials with s successes, following $\text{Binomial}(n, \theta)$

Posterior: $\theta|s \sim \text{Beta}(\alpha_0 + s, \beta_0 + n - s)$

After observing 20 trials with 8 successes:

$$\text{Prior mean:} \quad \frac{2}{2+8} = 0.2 \tag{5.18}$$

$$\text{Posterior mean:} \quad \frac{2+8}{2+8+8+12} = \frac{10}{30} = 0.33 \tag{5.19}$$

The posterior mean shifts toward the observed success rate of $8/20 = 0.4$.

5.7.2 Multi-Armed Bandit Problems

Definition 5.13 (Multi-Armed Bandit)**.** A sequential decision problem where:

- K arms (actions) with unknown reward distributions
- Each pull of arm i generates reward r_i from distribution F_i
- Goal: Maximize cumulative reward over time horizon
- Trade-off: Exploration (learning) vs. Exploitation (earning)

Example 5.8 (Two-Armed Bandit with Beta-Bernoulli Model)**.** **Setup:**

- Two arms with success probabilities θ_1, θ_2
- Priors: $\theta_i \sim \text{Beta}(\alpha_i, \beta_i)$
- Rewards: Success = 1, Failure = 0

Gittins Index Approach: For each arm i, compute the Gittins index $G_i(\alpha_i, \beta_i)$ which represents the value of playing that arm optimally forever vs. receiving a constant payoff.

Thompson Sampling Strategy:

```
for each time period t do
    for each arm i do
        Sample θ̃_i ~ Beta(α_i, β_i)
    end for
    Choose arm i* = argmax_i θ̃_i
    Observe reward r and update: (α_{i*}, β_{i*}) ← (α_{i*} + r, β_{i*} + 1 − r)
end for
```

Upper Confidence Bound (UCB) Strategy: Choose arm with highest upper confidence bound:

$$i^* = \text{argmax}_i \left[\hat{\mu}_i + c\sqrt{\frac{\log t}{n_i}} \right]$$

where $\hat{\mu}_i$ is empirical mean, n_i is number of times arm i was pulled, and c is confidence parameter.

5.8 Information Economics

5.8.1 Mechanism Design

Definition 5.14 (Mechanism Design Problem)**.** Design rules/procedures to achieve desired outcomes when participants have private information:

- Participants have private types θ_i
- Designer wants to implement social choice function $f(\theta_1, \ldots, \theta_n)$
- Must provide incentives for truthful revelation

Example 5.9 (Second-Price Auction)**. Mechanism:**

- Each bidder submits sealed bid
- Highest bidder wins
- Winner pays second-highest bid

Incentive Analysis: Suppose bidder i has valuation v_i and others' maximum bid is b_{-i}.
If $b_i > b_{-i}$ (bidder i wins):

- Payoff = $v_i - b_{-i}$
- Independent of b_i (as long as $b_i > b_{-i}$)

If $b_i < b_{-i}$ (bidder i loses):

- Payoff = 0
- Independent of b_i

Conclusion: Bidding $b_i = v_i$ (truthfully) is a dominant strategy.

Revenue Equivalence: Under certain conditions, all standard auction formats yield the same expected revenue.

5.8.2 Signaling Games

Definition 5.15 (Signaling Game)**.** A two-stage game where:

1. Informed player (Sender) observes private type and chooses signal
2. Uninformed player (Receiver) observes signal and chooses action
3. Payoffs depend on type, signal, and action

Case Study

Job Market Signaling (Spence Model):
Players:

- Workers have productivity $\theta \in \{\theta_L, \theta_H\}$ (private information)
- Employers observe education level e and set wages

Costs:

- Education cost for type θ: $c(e, \theta)$ with $c_e > 0, c_{e\theta} < 0$
- High-ability workers find education relatively less costly

Separating Equilibrium: Different types choose different education levels:

- $e_H > e_L$ (high types get more education)
- $w(e_H) = \theta_H, w(e_L) = \theta_L$ (wages equal productivity)

Incentive Compatibility:

$$\theta_H - c(e_H, \theta_H) \geq \theta_L - c(e_L, \theta_H) \quad \text{(High type doesn't mimic low)} \tag{5.20}$$
$$\theta_L - c(e_L, \theta_L) \geq \theta_H - c(e_H, \theta_L) \quad \text{(Low type doesn't mimic high)} \tag{5.21}$$

Welfare Implications:

- Education may be purely signaling (no productivity effect)
- Social waste: Resources spent on signaling rather than production
- But signaling can improve matching efficiency

Exercise 5.2. Consider a firm that can be either high-quality (θ_H) or low-quality (θ_L) with equal probability. The firm can choose advertising level $a \geq 0$ at cost $c(a, \theta) = a^2/(2\theta)$.

Consumers observe advertising and choose to buy or not. Their willingness to pay is θ for quality θ and 0 otherwise.

1. Find the separating equilibrium advertising levels 2. Calculate the welfare loss due to signaling 3. Compare with the pooling equilibrium where both types choose the same advertising level

Chapter 6

Robust Decision Making Under Deep Uncertainty

6.1 Introduction to Deep Uncertainty

Traditional decision analysis assumes well-defined probability distributions over uncertain events. However, many real-world problems involve **deep uncertainty** where:

Definition 6.1 (Deep Uncertainty)**.** Deep uncertainty exists when decision-makers do not know or cannot agree on:

- The appropriate models to describe interactions among system variables
- The probability distributions to represent uncertainty about key parameters
- How to value the desirability of alternative outcomes

Real-World Application

Climate Change Policy: Decisions about greenhouse gas reduction involve:

- Model uncertainty: How do economic and climate systems interact?
- Parameter uncertainty: What is climate sensitivity to CO ?
- Value uncertainty: How should we weigh costs today vs. future benefits?
- Deep uncertainty: Are our models fundamentally correct?

6.2 Robust Decision Making Framework

Definition 6.2 (Robust Decision Making (RDM))**.** RDM is an approach to decision making under deep uncertainty that:

1. Evaluates strategies across many plausible futures
2. Identifies robust strategies that perform well across scenarios
3. Uses computer-based scenario discovery to understand vulnerabilities
4. Employs iterative processes for strategy improvement

6.2.1 RDM Process

1: **Structure:** Define decision alternatives, uncertainties, and performance measures
2: **Evaluate:** Run model across large ensemble of scenarios
3: **Assess:** Identify robust strategies using robustness criteria
4: **Explore:** Use scenario discovery to understand strategy vulnerabilities
5: **Improve:** Design adaptive strategies that perform well across scenarios

Example 6.1 (Water Supply Planning Under Climate Uncertainty)**. Decision Problem:** City planning water infrastructure expansion

Strategies:

- S_1: Build new reservoir
- S_2: Implement conservation program
- S_3: Develop groundwater
- S_4: Hybrid approach

Uncertainties:

- Climate change magnitude
- Population growth rate
- Economic development patterns
- Technology costs

Performance Measures:

- Supply reliability
- Financial cost
- Environmental impact

Robustness Analysis: Evaluate each strategy across 10,000 scenarios combining different uncertainty realizations.

6.3 Robustness Criteria

6.3.1 Regret-Based Measures

Definition 6.3 (Maximum Regret)**.** For strategy s across scenario set Θ:

$$\text{MaxRegret}(s) = \max_{\theta \in \Theta} \left[\max_{s'} V(s', \theta) - V(s, \theta) \right]$$

A strategy is robust if it minimizes maximum regret.

Definition 6.4 (Percentile Regret)**.**

$$\text{Regret}_p(s) = p\text{-th percentile of regret distribution across scenarios}$$

Less conservative than maximum regret; focuses on typical rather than worst-case performance.

6.3.2 Satisficing Measures

Definition 6.5 (Satisficing Robustness)**.** For performance threshold τ:

$$\text{Robustness}(s) = \text{Fraction of scenarios where } V(s, \theta) \geq \tau$$

A strategy is robust if it achieves satisfactory performance in a high fraction of scenarios.

Example 6.2 (Comparing Robustness Measures)**.** Consider three strategies evaluated across 1000 scenarios:

Strategy	Mean Return	Std Dev	Max Regret	% Satisfactory ($\tau = 50$)
A	60	15	80	85%
B	55	8	40	95%
C	70	25	120	70%

Analysis:

- **Expected utility maximizer:** Prefers C (highest mean)
- **Minimax regret:** Prefers B (lowest maximum regret)
- **Satisficing:** Prefers B (highest reliability)

Choice depends on decision-maker's attitude toward risk and uncertainty.

6.4 Scenario Discovery

Definition 6.6 (Scenario Discovery)**.** Computational methods to identify combinations of uncertain factors that lead to strategy failure or success.

Goal: Find interpretable rules of the form "Strategy s fails when $X_1 \in [a, b]$ AND $X_2 > c$"

6.4.1 Patient Rule Induction Method (PRIM)

PRIM finds boxes in the uncertainty space that contain high concentrations of scenarios of interest.

```
1: Start with all scenarios
2: Repeat:
3: Find dimension and threshold that maximizes density of target scenarios
4: Create box by restricting that dimension
5: Remove scenarios outside box
6: Until: Stopping criterion met (e.g., minimum box size)
7: Peel: Remove restrictions to improve coverage while maintaining density
```

Detailed Solution

PRIM Example - Infrastructure Vulnerability:
Scenario: Identify conditions under which flood protection fails
Variables:

- Sea level rise: 0-100 cm
- Storm intensity: 1-5 scale

- Development density: Low/Medium/High
- Investment level: \$0-500M

PRIM Results: ”Flood protection fails in 85% of scenarios where:

- Sea level rise > 40 cm AND
- Storm intensity > 3 AND
- Investment < \$200M

Coverage: This rule captures 60% of all failure scenarios
Policy Implications:

- Investment threshold of \$200M appears critical
- System vulnerable to combination of factors, not individual extremes
- Can guide adaptive management triggers

6.5 Adaptive Decision Strategies

Definition 6.7 (Adaptive Strategy)**.** A decision strategy that:

- Monitors key indicators over time
- Includes pre-planned decision rules for strategy adjustment
- Maintains flexibility to change course as uncertainty resolves

6.5.1 Real Options Approach

Definition 6.8 (Real Option)**.** An investment that gives the right, but not obligation, to take future actions. Options have value because they provide flexibility under uncertainty.

Example 6.3 (Phased Infrastructure Development)**. Problem:** Airport expansion under uncertain demand growth

Traditional Approach: Build full capacity immediately

- Cost: \$500M upfront
- Risk: Overcapacity if demand is low

Real Options Approach: Phased expansion

- Phase 1: \$200M, handles current demand + 50%
- Option to expand: \$350M if demand materializes
- Monitor demand for 5 years before deciding

Option Value Calculation: Assume demand growth is either 5% (low) or 10% (high) annually with equal probability.

Traditional approach expected NPV:

$$NPV_{\text{traditional}} = 0.5 \times \text{NPV}_{\text{low demand}} + 0.5 \times \text{NPV}_{\text{high demand}}$$

Options approach expected NPV:

$$NPV_{\text{options}} = \text{NPV}_{\text{Phase 1}} + 0.5 \times \max(0, \text{NPV}_{\text{expansion}} - 350)$$

The option provides value through the ability to avoid the expansion cost in low-demand scenarios.

6.5.2 Dynamic Adaptive Policy Pathways

Definition 6.9 (Adaptive Pathways)**.** A planning approach that:

- Maps multiple possible routes toward objectives
- Identifies decision points and monitoring triggers
- Prepares contingent actions for different futures
- Enables learning and course correction

Case Study

Coastal Adaptation Pathways:
Objective: Protect coastal community from sea level rise
Pathway Elements:

- **Actions:** Beach nourishment, sea walls, managed retreat
- **Triggers:** Sea level thresholds, storm damage frequency
- **Decision Points:** Every 10 years or when triggers activated

Pathway Map:

1. **Now-2030:** Beach nourishment + monitoring
2. **2030 Decision Point:**
    - If SLR $<$ 20cm: Continue nourishment
    - If SLR 20-40cm: Add sea walls
    - If SLR $>$ 40cm: Begin managed retreat planning
3. **2040 Decision Point:** Further adaptation based on observed conditions

Advantages:

- Avoids premature commitment to expensive options
- Maintains flexibility as uncertainty resolves
- Builds stakeholder understanding of long-term choices

6.6 Multi-Objective Robust Optimization

Definition 6.10 (Robust Pareto Optimality)**.** A solution is robust Pareto optimal if there exists no other solution that dominates it across a significant portion of the uncertainty space.

Example 6.4 (Multi-Objective Portfolio Selection Under Model Uncertainty)**. Problem:** Select portfolio considering both return and risk, but uncertain about:

- True return distributions
- Correlation structures
- Model specification

Robust Approach:

1. Define uncertainty set $\mathcal{U}$ containing plausible models
2. For each candidate portfolio x:
   - Calculate performance across all models in $\mathcal{U}$
   - Compute robustness measures (e.g., worst-case return, regret)
3. Find portfolios that are robust Pareto optimal

Mathematical Formulation:

$$\min_{x} \max_{u \in \mathcal{U}} \left[-\mu(u)^T x, \quad x^T \Sigma(u) x \right]$$

where $\mu(u)$ and $\Sigma(u)$ represent return and covariance under model u.

Exercise 6.1. Consider a renewable energy investment decision under uncertainty about:

- Future electricity prices
- Technology costs
- Policy support (carbon pricing, subsidies)
- Demand growth

Design a robust decision-making analysis that includes:

1. Definition of alternative investment strategies
2. Specification of key uncertainties and their ranges
3. Selection of appropriate robustness criteria
4. Design of an adaptive strategy with decision triggers
5. Description of how scenario discovery could inform the analysis

Discuss how this approach differs from traditional financial analysis and when it might be preferable.

Chapter 7

Advanced Topics and Applications

7.1 Network Effects in Decision Making

Definition 7.1 (Network Externalities)**.** The value of a product or decision to one user depends on the number of other users making the same choice.

Types:

- **Direct:** Value increases with network size (e.g., telephone, social media)
- **Indirect:** Value increases through complementary products (e.g., operating systems, game consoles)

7.1.1 Adoption Decisions with Network Effects

Example 7.1 (Technology Adoption Game)**. Setup:** n consumers decide whether to adopt new technology

- Adoption cost: c
- Benefit if k others adopt: $b(k)$ with $b'(k) > 0$
- Net payoff from adoption: $b(k) - c$
- Payoff from non-adoption: 0

Equilibrium Conditions: Consumer adopts if $b(k) \geq c$, where k is expected number of other adopters.

Critical Mass: Minimum number of adopters k^* such that $b(k^*) = c$

Multiple Equilibria:

- No adoption: If fewer than k^* adopt, remaining consumers prefer not to adopt
- Full adoption: If more than k^* adopt, remaining consumers prefer to adopt
- Coordination problem: Multiple stable equilibria possible

Policy Implications:

- Temporary subsidies can overcome adoption barriers
- Early adopter incentives can trigger widespread adoption
- Standards setting can resolve coordination failures

7.2 Algorithmic Decision Making

7.2.1 Machine Learning in Decision Systems

Definition 7.2 (Algorithmic Decision Making)**.** Automated systems that use data and algorithms to make or recommend decisions affecting individuals or organizations.

Examples:

- Credit scoring and loan approval
- Resume screening and hiring
- Medical diagnosis assistance
- Criminal justice risk assessment

Exploration

Bias and Fairness in Algorithmic Decisions:
Sources of Bias:

- Historical bias in training data
- Measurement bias in feature selection
- Algorithmic bias in model design
- Evaluation bias in performance metrics

Fairness Criteria:

- **Individual fairness:** Similar individuals treated similarly
- **Group fairness:** Equal outcomes across groups
- **Counterfactual fairness:** Decisions unchanged in counterfactual world without sensitive attributes

Trade-offs: Different fairness criteria often conflict with each other and with accuracy.

7.2.2 Human-AI Collaboration

Example 7.2 (Medical Diagnosis with AI Assistance)**. Setup:** Doctor and AI system collaborate on diagnosis

Information Structure:

- Doctor observes patient symptoms and has medical knowledge
- AI analyzes medical images and provides probability assessments
- Both have imperfect information

Decision Framework:

1. AI provides diagnostic probabilities: $P(\text{Disease}|\text{Image})$

2. Doctor updates beliefs using clinical information
3. Combined assessment guides treatment decisions

Optimal Integration: Weight AI and human inputs based on:

- Relative accuracy in different contexts
- Complementarity of information sources
- Bias and overconfidence patterns

Challenges:

- Automation bias: Over-reliance on algorithmic recommendations
- Algorithm aversion: Systematic under-utilization of AI insights
- Calibration: Ensuring probability assessments are well-calibrated

7.3 Collective Decision Making

7.3.1 Social Choice Theory

Definition 7.3 (Social Choice Function)**.** A mapping from individual preferences to social rankings or choices.

Desirable Properties:

- **Unanimity:** If everyone prefers x to y, so does society
- **Independence:** Social ranking of x vs y depends only on individual rankings of x vs y
- **Non-dictatorship:** No single individual determines all social choices

Theorem

Arrow's Impossibility Theorem: No social choice function satisfies unanimity, independence, and non-dictatorship simultaneously when there are at least 3 alternatives and 2 individuals.

Example 7.3 (Voting Paradox)**.** Three voters rank three alternatives:

	Voter 1	Voter 2	Voter 3
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

Pairwise Comparisons:

- A vs B: A wins (Voters 1,3 prefer A)
- B vs C: B wins (Voters 1,2 prefer B)
- C vs A: C wins (Voters 2,3 prefer C)

Result: Intransitive social preference: A B C A

This demonstrates the impossibility of aggregating individual preferences into coherent social rankings.

7.3.2 Deliberation and Information Aggregation

Definition 7.4 (Condorcet Jury Theorem)**.** If individuals make independent judgments with probability $p > 0.5$ of being correct, then as group size increases, the probability that majority vote is correct approaches 1.

Example 7.4 (Information Cascades in Committee Decisions)**. Setup:** Committee members decide sequentially, observing predecessors' choices

Process:

1. Each member receives private signal about best decision
2. Members vote in sequence, observing all previous votes
3. Each wants to make correct decision

Cascade Formation:

- Early voters may follow predecessors despite conflicting private information
- Later voters ignore their private signals
- Committee may converge on wrong decision

Welfare Implications:

- Information is not efficiently aggregated
- Society makes worse decisions than if all private information were public
- Design interventions: simultaneous voting, devil's advocate roles

7.4 Experimental Decision Theory

7.4.1 Laboratory Experiments

Case Study

Testing Expected Utility Theory:
Allais Paradox Experiment:
Choice Set 1:

- Option A: $1M with certainty
- Option B: $5M with prob 0.1, $1M with prob 0.89, $0 with prob 0.01

Choice Set 2:

- Option C: $1M with prob 0.11, $0 with prob 0.89
- Option D: $5M with prob 0.1, $0 with prob 0.9

Typical Results:

- Most people choose A over B

- Most people choose D over C

Expected Utility Prediction: If A B, then EU theory requires C D (by independence axiom)
Interpretation: Systematic violation of independence axiom supports prospect theory over expected utility theory.

7.4.2 Field Experiments

Real-World Application

Nudging Retirement Savings:
Experimental Design:

- Treatment: Automatic enrollment in 401(k) with opt-out option
- Control: Standard opt-in enrollment
- Outcome: Participation rates after 1 year

Results:

- Control group: 20% participation
- Treatment group: 85% participation
- Effect persists over time (limited opt-out)

Behavioral Mechanisms:

- Status quo bias: People stick with default
- Procrastination: Opt-in requires active decision
- Implied endorsement: Default suggests recommended action

Policy Applications:

- Widespread adoption of automatic enrollment
- Extension to contribution rates and investment choices
- Application to health insurance, organ donation

7.5 Future Directions

7.5.1 Artificial Intelligence and Decision Theory

Exploration

AI Decision Systems:
Emerging Challenges:

- Multi-agent AI systems with strategic interactions
- AI systems that learn and adapt decision rules
- Human oversight of algorithmic decisions
- Explainable AI for decision support

Research Frontiers:

- Mechanism design for AI agents
- Robust AI decision making under distributional shift
- AI safety and alignment with human values
- Integration of symbolic and neural approaches

7.5.2 Behavioral Insights and Technology

Definition 7.5 (Digital Nudging)**.** Use of user interface design elements to guide behavior in digital environments.

Examples:

- Default privacy settings
- Framing of online choices
- Social comparison information
- Timing of notifications

Exercise 7.1. Comprehensive Decision Analysis Project:

Choose a real-world decision problem (e.g., urban planning, healthcare policy, business strategy) and develop a comprehensive analysis that incorporates:

1. **Problem Structuring:**
   - Define decision alternatives
   - Identify key uncertainties
   - Specify stakeholders and their objectives
2. **Multiple Decision Frameworks:**
   - Traditional expected utility analysis

   - Behavioral considerations (biases, framing effects)
   - Game-theoretic interactions if relevant
   - Multi-criteria analysis for multiple objectives

3. **Uncertainty Treatment:**
   - Probability assessment methods
   - Robust decision making for deep uncertainty
   - Value of information analysis

4. **Implementation Considerations:**
   - Dynamic and adaptive strategies
   - Stakeholder engagement process
   - Monitoring and evaluation framework

Prepare a report that includes mathematical analysis, behavioral insights, and practical recommendations. Discuss how different theoretical frameworks lead to different conclusions and how to reconcile these differences in practice.

Appendix A

Mathematical Foundations

A.1 Probability Theory Review

Definition A.1 (Probability Space)**.** A probability space consists of:

- Sample space Ω (set of all possible outcomes)
- σ-algebra $\mathcal{F}$ (collection of events)
- Probability measure $P : \mathcal{F} \to [0, 1]$

A.2 Optimization Theory

Theorem A.1 (Kuhn-Tucker Conditions)**.** *For the problem* $\max f(x)$ *subject to* $g_i(x) \leq 0$ *and* $x \geq 0$*:*

If x^* *is optimal, then there exist* $\lambda_i \geq 0$ *such that:*

$$\nabla f(x^*) - \sum_i \lambda_i \nabla g_i(x^*) \leq 0 \tag{A.1}$$

$$x^*[\nabla f(x^*) - \sum_i \lambda_i \nabla g_i(x^*)] = 0 \tag{A.2}$$

$$\lambda_i g_i(x^*) = 0 \tag{A.3}$$

$$g_i(x^*) \leq 0 \tag{A.4}$$

Appendix B

Software Tools

B.1 Decision Analysis Software

- **TreeAge Pro:** Decision trees and Markov models
- **@RISK:** Monte Carlo simulation in Excel
- **DecisionTools Suite:** Comprehensive decision analysis
- **R packages:** MCMCpack, rjags, bnlearn
- **Python libraries:** pgmpy, pomegranate, scikit-learn

B.2 Game Theory Software

- **Gambit:** Game theory analysis tools
- **MATLAB Game Theory Toolbox**
- **Python libraries:** nashpy, pygambit