Lecture Notes of Mathematical Modeling

Chapter 9: Decision Theory and Game Theory

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Chapter Overview

Learning Objectives

By the end of this chapter, you will be able to

- Understand mathematical foundations of decision theory under uncertainty
- Formulate and solve complex decision problems using Bayesian analysis
- Analyze strategic interactions using game theory and Nash equilibria
- Apply evolutionary stability concepts and mechanism design
- Implement computational algorithms for finding equilibria
- Understand auction theory and resource allocation applications
- Design optimal strategies for multi-agent systems

Why Decision and Game Theory?

The Challenge of Strategic Thinking

Decision theory and game theory provide mathematical frameworks for analyzing choice under uncertainty and strategic interaction among rational agents.

Revolutionary impact across fields:

- Economics: Market analysis and policy design
- Biology: Evolutionary dynamics and behavior
- Computer Science: Algorithm design and AI
- Political Science: Voting and conflict analysis

Why Decision and Game Theory?

Modern Applications:

- **Auction Design**: Spectrum allocation worth billions of dollars
- **Evolutionary Biology**: Understanding sex ratios and cooperation
- **Autonomous Systems**: Coordination of self-driving vehicles
- **Cybersecurity**: Strategic defense against adversaries

Why Decision and Game Theory?

Mathematical Rigor

These theories enable precise analysis of seemingly intractable problems involving strategic behavior, uncertainty, and conflicting interests.

Decision Problems Under Uncertainty

Definition (Decision Problem Under Uncertainty)

A decision problem consists of:

1. Actions $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ available to decision-maker
2. States of nature $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$
3. Consequence function $c : \mathcal{A} \times \Theta \to \mathcal{X}$
4. Probability distribution $P(\theta)$ over states (if known)

Decision Problems Under Uncertainty

The Central Challenge:

- Cannot perfectly predict which state will occur
- Must choose action before uncertainty resolves
- Different actions perform better under different states
- Need systematic approach for optimal choice

Decision Problems Under Uncertainty

Examples:

- **Medical Treatment**: Choose therapy before knowing patient response
- **Investment Decisions**: Portfolio allocation under market uncertainty
- **Business Strategy**: Product launch with uncertain demand
- **Policy Making**: Intervention with uncertain outcomes

Expected Utility Theory

Theorem (Expected Utility Theorem)

If preferences over lotteries satisfy axioms of completeness, transitivity, continuity, and independence, then there exists utility function $u : \mathcal{X} \to \mathbb{R}$ such that:

$$L_1 \succeq L_2 \iff \mathbb{E}[u(L_1)] \geq \mathbb{E}[u(L_2)]$$

Expected Utility Theory

Key Axioms:

- **Completeness**: Can compare any two lotteries
- **Transitivity**: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- **Continuity**: Preferences don't have sudden jumps
- **Independence**: Preferences over outcomes independent of irrelevant alternatives

Expected Utility Theory

Practical Implications:

- Rational choice = maximize expected utility
- Utility function captures risk preferences
- Provides normative foundation for decision-making
- Enables quantitative analysis of complex decisions

Power of Formalization

Transforms intuitive decision-making into rigorous mathematical optimization.

Risk Preferences and Utility Functions

Definition (Risk Aversion Measures)

For twice-differentiable utility function u:

$$\text{Absolute Risk Aversion:} \quad A(x) = -\frac{u''(x)}{u'(x)} \tag{1}$$

$$\text{Relative Risk Aversion:} \quad R(x) = -\frac{xu''(x)}{u'(x)} \tag{2}$$

Risk Preferences and Utility Functions

Risk Attitude Classification:

- **Risk Averse**: $u''(x) < 0$ (concave utility)
- **Risk Neutral**: $u''(x) = 0$ (linear utility)
- **Risk Seeking**: $u''(x) > 0$ (convex utility)

Intuition: Diminishing marginal utility of wealth leads to risk aversion.

Risk Preferences and Utility Functions

Practical Applications:

- Insurance demand and optimal coverage
- Portfolio optimization and asset allocation
- Pricing of financial derivatives
- Public policy evaluation under uncertainty

Real-World Relevance

Risk preferences explain why people buy insurance even when premiums exceed expected losses.

Bayesian Framework for Information

Definition (Bayesian Decision Problem)

A Bayesian decision problem includes:

1. Prior beliefs $P(\theta)$ over states
2. Information sources with likelihood functions $P(s|\theta)$
3. Posterior beliefs via Bayes' rule: $P(\theta|s) = \frac{P(s|\theta)P(\theta)}{P(s)}$
4. Decision rules mapping information to actions

Bayesian Framework for Information

Value of Information:
Expected Value of Perfect Information (EVPI):

$$EVPI = \sum_{\theta} P(\theta) \max_{a} u(a,\theta) - \max_{a} \sum_{\theta} P(\theta) u(a,\theta)$$

Expected Value of Sample Information (EVSI):

$$EVSI = \sum_{s} P(s) \max_{a} \sum_{\theta} P(\theta|s) u(a,\theta) - \max_{a} \sum_{\theta} P(\theta) u(a,\theta)$$

Bayesian Framework for Information

Key Insights:

- Information has value only if it changes decisions
- Perfect information provides upper bound on information value
- Sample information value depends on accuracy and relevance
- Information gathering should be compared to its cost

Applications: Medical testing, market research, scientific experimentation, intelligence gathering.

Medical Diagnosis Example

Example (Diagnostic Testing Decision)

Physician deciding whether to order expensive test:

- Disease prevalence: $P(D) = 0.1$
- Test sensitivity: 0.9, specificity: 0.95
- Treatment utilities vary by disease state

Medical Diagnosis Example

Without Testing:

$$\mathbb{E}[u(\text{Treat})] = 0.1 \times 0.9 + 0.9 \times 0.7 = 0.72 \quad (1)$$

$$\mathbb{E}[u(\text{Don't Treat})] = 0.1 \times 0.1 + 0.9 \times 1.0 = 0.91 \quad (2)$$

Optimal action: Don't treat (expected utility = 0.91)

Medical Diagnosis Example

With Testing: Update beliefs using test results
Posterior probabilities:

$$P(D|+) = \frac{0.9 \times 0.1}{0.9 \times 0.1 + 0.05 \times 0.9} = 0.67 \quad (1)$$

$$P(D|-) = \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.95 \times 0.9} = 0.012 \quad (2)$$

Expected utility with testing: 0.977
Value of testing: $0.977 - 0.91 = 0.067$ utility units

Strategic Form Games

Definition (Strategic Form Game)

A game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ consists of:

1. Players $N = \{1, 2, \ldots, n\}$
2. Strategy sets S_i for each player i
3. Payoff functions $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$

Strategic Form Games

Key Features:

- Simultaneous decision-making
- Each player's payoff depends on all players' choices
- Strategic interdependence creates complex dynamics
- Mathematical framework enables systematic analysis

Strategic Form Games

Classic Examples:

- **Prisoner's Dilemma**: Cooperation vs. defection
- **Coordination Games**: Multiple equilibria, coordination problems
- **Battle of Sexes**: Conflicting preferences with coordination benefits
- **Zero-Sum Games**: Pure conflict situations

These simple games capture essential strategic features of complex real-world interactions.

Nash Equilibrium

Definition (Nash Equilibrium)

Strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is Nash equilibrium if for every player i:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all strategies $s_i \in S_i$.

Nash Equilibrium

Theorem (Nash Equilibrium Existence)

Every finite strategic form game has at least one Nash equilibrium in mixed strategies.

Proof idea: Uses Brouwer fixed point theorem applied to best response correspondences.

Nash Equilibrium

Interpretation:

- No player can unilaterally improve their payoff
- Self-enforcing: no incentive to deviate
- Prediction of rational play
- Stable outcome of strategic interaction

Fundamental Concept

Nash equilibrium provides the primary solution concept for non-cooperative games.

Mixed Strategy Equilibria

Battle of the Sexes Game:

	Opera	Football
Opera	(2,1)	(0,0)
Football	(0,0)	(1,2)

Three Nash equilibria: (Opera, Opera), (Football, Football), and one mixed.

Mixed Strategy Equilibria

Theorem (Indifference Principle)

In mixed strategy Nash equilibrium, players must be indifferent among all strategies in their support.

For Battle of Sexes mixed equilibrium:

- Player 1 chooses Opera with probability $2/3$
- Player 2 chooses Opera with probability $1/3$
- Expected payoffs: $(2/3, 2/3)$ for both players

Mixed Strategy Equilibria

Why Mixed Strategies?

- Pure strategies may not yield equilibrium
- Randomization can be optimal response to opponent's randomization
- Common in competitive situations (sports, military, cybersecurity)
- Mathematical guarantee of equilibrium existence

Evolutionary Stability

Definition (Evolutionarily Stable Strategy (ESS))

Strategy s^* is evolutionarily stable if for any alternative strategy $s \neq s^*$, there exists $\bar{\epsilon} > 0$ such that for $0 < \epsilon < \bar{\epsilon}$:

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

Evolutionary Stability

ESS Conditions:

1. $u(s^*, s^*) \geq u(s, s^*)$ for all s (Nash condition)
2. If $u(s, s^*) = u(s^*, s^*)$, then $u(s^*, s) > u(s, s)$

Intuition: ESS is uninvadable by small groups of mutants.

Evolutionary Stability

Definition (Replicator Dynamics)

Evolution of strategy frequencies:

$$\frac{d}{dt}x_i(t) = x_i(t)[f_i(x(t)) - \bar{f}(x(t))]$$

where f_i is fitness of strategy i, $\bar{f}$ is average fitness.

Properties: ESS are asymptotically stable under replicator dynamics.

Hawk-Dove Game

Example (Animal Conflict Model)

Two strategies: Hawk (aggressive), Dove (peaceful)
Payoff matrix with resource value $V = 10$, fighting cost $C = 15$:

	Hawk	Dove
Hawk	$\frac{V-C}{2} = -2.5$	$V = 10$
Dove	0	$\frac{V}{2} = 5$

Hawk-Dove Game

ESS Analysis:

- Pure Hawk: Not ESS (negative payoff against itself)
- Pure Dove: Not ESS (invaded by Hawks)
- Mixed ESS: Proportion of Hawks $= V/C = 10/15 = 2/3$

Hawk-Dove Game

Biological Interpretation:

- Explains why animals don't always fight to the death
- Frequency-dependent selection maintains polymorphism
- Cost-benefit analysis determines equilibrium aggression levels
- Foundation for understanding animal behavior evolution

Insight

Even purely selfish behavior can lead to moderated aggression through evolutionary dynamics.

Mechanism Design: Reverse Game Theory

Definition (Mechanism)

A mechanism consists of:

1. Message spaces M_i for each agent i
2. Outcome function $g : M_1 \times \cdots \times M_n \to A$
3. Payment functions $t_i : M_1 \times \cdots \times M_n \to \mathbb{R}$

Goal: Design games to achieve desired outcomes.

Mechanism Design: Reverse Game Theory

Desirable Properties:

- **Incentive Compatibility**: Truth-telling is optimal
- **Individual Rationality**: Participation is beneficial
- **Efficiency**: Maximizes social welfare
- **Revenue Maximization**: Maximizes designer's revenue

Mechanism Design: Reverse Game Theory

Applications:

- Auction design for spectrum allocation
- Voting systems and preference aggregation
- Contract theory and organizational design
- Algorithmic mechanism design for computer systems

Challenge: These properties often conflict, requiring careful trade-offs.

Auction Theory

Common Auction Formats:

- **First-Price Sealed-Bid**: Highest bidder wins, pays their bid
- **Second-Price Sealed-Bid**: Highest bidder wins, pays second-highest bid
- **English Auction**: Open ascending price auction
- **Dutch Auction**: Open descending price auction

Auction Theory

Theorem (Revenue Equivalence Theorem)

Under standard conditions (independent private values, risk neutrality, efficient allocation), all auction formats yield the same expected revenue.

Conditions:

- Highest bidder wins
- Lowest-value bidder has zero expected payment
- Same information structure across formats

Auction Theory

Strategic Differences:

- **Second-Price**: Truth-telling is dominant strategy
- **First-Price**: Bid shading optimal (bid below value)
- **English**: Efficient but reveals information
- **Dutch**: Strategically equivalent to first-price

Practical Impact

Revenue equivalence means auction choice should consider other factors: simplicity, transparency, collusion resistance.

Economic Applications

Oligopoly Competition:
Cournot Model (quantity competition):

- Firms choose quantities simultaneously
- Market price determined by total quantity
- Strategic substitutes: higher competitor quantity → lower own quantity

Bertrand Model (price competition):

- Firms choose prices simultaneously
- With identical products: price equals marginal cost
- Strategic complements: higher competitor price → higher own price

Economic Applications

Auction Applications:

- **Spectrum Auctions**: Generated over $\$100$ billion in government revenue
- **Electricity Markets**: Real-time bidding for power generation
- **Online Advertising**: Ad space allocation through automated auctions
- **Treasury Securities**: Government debt issuance

Economic Applications

Contract Theory:

- Principal-agent problems with moral hazard
- Optimal incentive design in organizations
- Insurance contracts with adverse selection
- Executive compensation design

These applications demonstrate how game theory guides real-world institutional design.

Biological Applications

Sex Ratio Evolution (Fisher's Principle):
Population with proportion p males. Fitness of strategy producing proportion x males:

$$f(x,p) = \frac{x}{p} + \frac{1-x}{1-p}$$

ESS condition: $\frac{\partial f}{\partial x} = \frac{1}{p} - \frac{1}{1-p} = 0$
Solution: $p = 1/2$ (equal sex ratio)

Biological Applications

Other Biological Applications:

- **Foraging Behavior**: Optimal patch selection strategies
- **Cooperation Evolution**: Explaining altruism and reciprocity
- **Signaling Systems**: Honest vs. deceptive communication
- **Territorial Behavior**: Space and resource competition

Biological Applications

Human Behavior:

- **Social Norms**: Evolution of cooperation and punishment
- **Language Evolution**: Communication system development
- **Cultural Evolution**: Transmission of behaviors and beliefs
- **Conflict Resolution**: Understanding war and peace

Game theory bridges social and biological sciences through unified mathematical framework.

Computing Equilibria

Theorem (PPAD-Completeness)

Computing Nash equilibrium in two-player games is PPAD-complete, even with two strategies per player.

Implications:

- No polynomial-time algorithm expected
- Problem belongs to class with guaranteed solutions
- Computational difficulty despite existence guarantee

Computing Equilibria

Algorithms for Equilibrium Computation:

- **Lemke-Howson**: Pivot algorithm for two-player games
- **Support Enumeration**: Check all possible supports
- **Evolutionary Algorithms**: Replicator dynamics simulation
- **Fictitious Play**: Iterative best-response learning

Computing Equilibria

Modern Developments:

- **Approximate Equilibria**: Relaxed solution concepts
- **Large-Scale Games**: Algorithms for massive games
- **Online Learning**: Adapting to changing environments
- **Multi-Agent Reinforcement Learning**: AI applications

These advances enable game-theoretic analysis of previously intractable problems.

Theoretical Foundations

What We've Accomplished:

- Expected utility theory for rational decision-making
- Bayesian analysis for information and learning
- Nash equilibrium for strategic interactions
- Evolutionary stability for dynamic systems
- Mechanism design for institutional engineering
- Auction theory for resource allocation

Theoretical Foundations

Mathematical Unity:

- Fixed point theorems underlie equilibrium existence
- Optimization principles guide solution concepts
- Probability theory enables uncertainty analysis
- Dynamic systems theory explains evolution and learning

Theoretical Foundations

Computational Reality:

- Computational complexity limits exact solutions
- Approximation algorithms enable practical applications
- Simulation methods explore complex dynamics
- Machine learning approaches automate strategy discovery

Integration

Theory, computation, and applications form unified framework for strategic analysis.

Practical Impact Across Domains

Economics and Finance:

- Market design and auction mechanisms
- Risk management and portfolio theory
- Industrial organization and competition policy
- Behavioral economics and bounded rationality

Practical Impact Across Domains

Technology and Computing:

- Internet protocols and network design
- Cryptocurrency and blockchain mechanisms
- Artificial intelligence and multi-agent systems
- Cybersecurity and adversarial settings

Practical Impact Across Domains

Social Sciences and Policy:

- Voting systems and democratic institutions
- International relations and conflict resolution
- Environmental policy and climate agreements
- Public health interventions and compliance

These applications demonstrate the broad relevance of mathematical frameworks for strategic thinking.

Future Directions

Emerging Frontiers:

- **Behavioral Game Theory**: Incorporating psychological insights
- **Algorithmic Mechanism Design**: Computer science applications
- **Network Games**: Strategic interactions on graphs
- **Quantum Game Theory**: Quantum mechanical strategies

Future Directions

Technological Applications:

- Autonomous vehicle coordination
- Smart grid optimization
- Social media and information networks
- Distributed computing systems

Future Directions

Societal Challenges:

- Climate change coordination
- Pandemic response strategies
- Digital privacy and surveillance
- Artificial intelligence governance

Continuing Evolution

Mathematical frameworks continue expanding to address emerging strategic challenges in our interconnected world.

Key Takeaways

- Decision theory provides normative foundations for rational choice under uncertainty
- Game theory analyzes strategic interactions with mathematical precision
- Nash equilibrium offers fundamental solution concept for strategic stability
- Evolutionary approaches explain dynamics and long-run behavior
- Mechanism design enables engineering of strategic environments
- Computational methods make complex strategic analysis practical
- Applications span economics, biology, computer science, and social policy

These mathematical frameworks transform strategic thinking from intuition and experience into systematic, rigorous analysis.

Thank You

Questions and Discussion

Mathematical frameworks for strategic decision-making under uncertainty

Course Conclusion:

Mathematical Modeling: From Theory to Practice

Integrating all techniques for comprehensive problem-solving