Lecture Notes for Mathematical Modeling

Chapter 7: Discrete Probabilistic Modeling

Kenneth, Sok Kin Cheng

Chapter Overview

Learning Objectives

By the end of this chapter, you will be able to

- Construct and analyze Markov chain models for state-dependent transitions
- Apply queueing theory to optimize service systems and resource allocation
- Develop reliability models for complex systems with multiple failure modes
- Use decision theory frameworks for optimal choices under uncertainty
- Implement Monte Carlo methods for discrete stochastic processes
- Analyze game-theoretic models of strategic interactions
- Integrate probabilistic models with optimization techniques

Why Discrete Probabilistic Models?

The Power of Probabilistic Thinking

Discrete probabilistic modeling provides mathematical frameworks for analyzing systems where outcomes are uncertain and states change according to probabilistic rules.

Essential for understanding:

- Financial market dynamics and risk management
- Epidemic disease spread and intervention strategies
- Network reliability and communication protocols
- Strategic decision-making under uncertainty

Why Discrete Probabilistic Models?

Modern Applications:

- **Operations Research**: Queueing models optimize service systems
- **Finance**: Stochastic models price derivatives and manage risk
- **Epidemiology**: Markov models track disease progression
- **Artificial Intelligence**: Probabilistic graphical models enable machine learning

Why Discrete Probabilistic Models?

Key Insight

The power lies in capturing essential randomness while maintaining mathematical tractability for analysis and optimization.

Markov Chains: The Memoryless Property

Definition (Discrete-Time Markov Chain)

A sequence of random variables $\{X_n : n = 0, 1, 2, \ldots\}$ taking values in state space $\mathcal{S}$ such that:

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j | X_n = i) = p_{ij}$$

Markov Chains: The Memoryless Property

Key Properties:

- **Memoryless**: Future depends only on present, not history
- **Transition matrix**: $\mathbf{P}$ where each row sums to unity
- **Stochastic evolution**: Complete probabilistic description

Why this matters:

- Enables powerful mathematical analysis
- Captures important classes of real phenomena
- Foundation for computational algorithms

Chapman-Kolmogorov Equations

Theorem (Chapman-Kolmogorov Equations)

For a Markov chain with transition matrix $\mathbf{P}$*, the* n*-step transition probabilities satisfy:*

$$p_{ij}^{(n)} = \sum_{k \in \mathcal{S}} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

In matrix form: $\mathbf{P}^{(n)} = \mathbf{P}^n$

Chapman-Kolmogorov Equations

Intuitive Explanation:

- To go from state i to state j in n steps
- Must pass through some intermediate state k at step m
- Probability $=$ (probability to reach k) $\times$ (probability from k to j)
- Sum over all possible intermediate states

Chapman-Kolmogorov Equations

Mathematical Power:

- Matrix multiplication captures multi-step transitions
- $\mathbf{P}^n$ gives all n-step probabilities
- Enables analysis of long-term behavior
- Foundation for computing stationary distributions

Stationary Distributions

Definition (Stationary Distribution)

A probability distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$ is stationary if:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

Equivalently: $\pi_j = \sum_i \pi_i p_{ij}$ for all states j.

Stationary Distributions

Theorem (Fundamental Theorem for Finite Irreducible Markov Chains)

For a finite, irreducible, and aperiodic Markov chain:

1. *Unique stationary distribution* $\boldsymbol{\pi}$ *exists*
2. $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ *for all states* i, j
3. $\pi_j = \frac{1}{\mathbb{E}[T_j]}$ *where* $\mathbb{E}[T_j]$ *is expected return time*

Stationary Distributions

Practical Implications:

- Long-run proportion of time in each state
- Independent of starting state
- Enables steady-state analysis
- Foundation for optimization and control

Computing Stationary Distribution

Solve $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ subject to $\sum_i \pi_i = 1$

Application: Social Network Information Cascade

Example (Information Spread Model)

Four states: Unaware (U), Exposed (E), Adopter (A), Abandoner (B)
Transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.5 & 0.1 \\ 0.0 & 0.0 & 0.8 & 0.2 \\ 0.1 & 0.0 & 0.0 & 0.9 \end{pmatrix}$$

Application: Social Network Information Cascade

Solving for Stationary Distribution:

$$\pi_U = 0.7\pi_U + 0.1\pi_B \tag{1}$$

$$\pi_E = 0.3\pi_U + 0.4\pi_E \tag{2}$$

$$\pi_A = 0.5\pi_E + 0.8\pi_A \tag{3}$$

$$\pi_B = 0.1\pi_E + 0.2\pi_A + 0.9\pi_B \tag{4}$$

Application: Social Network Information Cascade

Solution: $\boldsymbol{\pi} = (0.1, 0.15, 0.375, 0.375)$
Interpretation:

- 37.5% eventual adopters
- 37.5% eventual abandoners
- Only 10% remain unaware
- 15% in exposed state

This reveals long-term information cascade dynamics in social networks.

Queueing Systems: Mathematical Structure

Definition (Queueing System Notation)

Kendall notation $A/B/c/N/K/D$ where:

$$\begin{aligned} A &: \text{Arrival process (M=Markovian, D=Deterministic, G=General)} \\ B &: \text{Service time distribution} \\ c &: \text{Number of servers} \\ N &: \text{System capacity (default: } \infty) \\ K &: \text{Customer population (default: } \infty) \\ D &: \text{Service discipline (default: FIFO)} \end{aligned}$$

Queueing Systems: Mathematical Structure

Key Performance Measures:

- L = Expected number of customers in system
- L_q = Expected number in queue (waiting)
- W = Expected time in system
- W_q = Expected waiting time
- ρ = System utilization

Queueing Systems: Mathematical Structure

Why Queueing Theory Matters:

- Optimize resource allocation in service systems
- Balance service capacity with customer satisfaction
- Minimize costs while maintaining quality
- Enable quantitative analysis of trade-offs

Little's Law: A Fundamental Result

Theorem (Little's Law)

For any stable queueing system in steady state:

$$L = \lambda W$$

where L = expected number in system, λ = arrival rate, W = expected time in system.

Little's Law: A Fundamental Result

Intuitive Explanation:

- Consider customers arriving and departing over time
- Total customer-time = (number of customers) $\times$ (time each spends)
- This equals integral of customers in system over time
- Dividing by time period gives the relationship

Little's Law: A Fundamental Result

Remarkable Properties:

- Holds for ANY stable queueing system
- Independent of arrival process, service distribution, queue discipline
- Connects time averages with customer averages
- Enables calculation of unknown measures from known ones

Universal Applicability

Little's Law applies to any system with arrivals and departures: hospitals, computer systems, manufacturing, traffic flow.

M/M/1 Queue: The Foundation

Theorem (M/M/1 Queue Performance)

For arrival rate λ, service rate μ, with $\rho = \lambda/\mu < 1$:

$$P_n = (1-\rho)\rho^n \quad \text{(probability of } n \text{ customers)} \tag{1}$$

$$L = \frac{\rho}{1-\rho} \quad \text{(expected number in system)} \tag{2}$$

$$W = \frac{1}{\mu-\lambda} \quad \text{(expected time in system)} \tag{3}$$

M/M/1 Queue: The Foundation

Key Insights:

- Higher utilization $\rightarrow$ exponentially longer waits
- Performance degrades rapidly as $\rho \rightarrow 1$
- Trade-off between efficiency and service quality
- Mathematical precision guides capacity planning

M/M/1 Queue: The Foundation

Example (Hospital Emergency Department)

Arrivals: $\lambda = 8$ patients/hour, Service: $\mu = 10$ patients/hour
$\rho = 0.8$, so:

- $L = 4$ patients in system
- $W = 30$ minutes total time
- $W_q = 24$ minutes waiting time

Adding second physician dramatically improves performance.

Reliability: Quantifying System Performance

Definition (Reliability Function)

The reliability function $R(t)$ is the probability that a system operates without failure for time duration t:

$$R(t) = P(T > t)$$

where T is the random time to failure.

The failure rate function: $\lambda(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt} \ln R(t)$

Reliability: Quantifying System Performance

System Architectures:

Series System (fails if any component fails):

$$R_{\text{series}}(t) = \prod_{i=1}^{n} R_i(t)$$

Parallel System (fails only if all components fail):

$$R_{\text{parallel}}(t) = 1 - \prod_{i=1}^{n} [1 - R_i(t)]$$

Reliability: Quantifying System Performance

Design Implications:

- Series systems: Reliability decreases with more components
- Parallel systems: Redundancy dramatically improves reliability
- Critical for safety-critical systems
- Guides maintenance and design decisions

Exponential Components

For exponential components with rates λ_i: - Series: $R(t) = \exp(-t \sum_i \lambda_i)$ - Parallel: More complex but much better

Markovian Reliability Models

Example (Data Center Power Systems)

Four states:

- State 0: Both power systems operational
- State 1: Primary failed, backup operational
- State 2: Backup failed, primary operational
- State 3: Both failed (system failure)

Markovian Reliability Models

Generator Matrix:

$$\mathbf{Q} = \begin{pmatrix} -2\lambda & \lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & 0 & \lambda \\ \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & \mu & \mu & -2\mu \end{pmatrix}$$

where $\lambda =$ failure rate, $\mu =$ repair rate.

Markovian Reliability Models

System Reliability: $R(t) = P_0(t) + P_1(t) + P_2(t)$
Advantages of Markovian Models:

- Capture state-dependent failure/repair rates
- Model complex interactions between components
- Enable optimization of maintenance strategies
- Provide detailed transient and steady-state analysis

Decision Theory Framework

Definition (Decision Problem)

A decision problem consists of:

1. Actions $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$
2. States of nature $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$
3. Loss function $L(a, \theta)$
4. Prior distribution $\pi(\theta)$ over states

Decision Theory Framework

Theorem (Bayes Decision Rule)

The optimal decision minimizes expected loss:

$$a^* = \arg\min_{a \in \mathcal{A}} \sum_{\theta} L(a, \theta) \pi(\theta)$$

With sample information x:

$$a^*(x) = \arg\min_{a \in \mathcal{A}} \sum_{\theta} L(a, \theta) \pi(\theta | x)$$

Decision Theory Framework

Value of Information:

- Expected Value of Perfect Information (EVPI)
- Expected Value of Sample Information (EVSI)
- Guides information gathering decisions
- Quantifies worth of reducing uncertainty

Optimality

Bayes rule achieves minimum possible expected loss when prior correctly represents beliefs.

Investment Decision Example

Example (Investment Under Uncertainty)

Actions: Invest in A, Invest in B, Don't invest States: Market up, stable, down
Payoff matrix:

$$\begin{pmatrix} 100 & 20 & -50 \\ 30 & 40 & -10 \\ 0 & 0 & 0 \end{pmatrix}$$

Prior: $(0.3, 0.5, 0.2)$

Investment Decision Example

Without Information: Expected payoffs: $(29, 27, 0)$ Best action: Invest in A
With Market Research: - Update beliefs using Bayes' theorem - Make conditional decisions based on signals - Expected value increases due to better information

Investment Decision Example

Information Analysis:

- EVSI = Value of market research
- Compare cost of research to expected benefit
- Optimal information gathering strategy
- Foundation for adaptive decision-making

This framework applies to medical diagnosis, financial planning, engineering design, and strategic planning.

Game Theory: Strategic Decision Making

Definition (Normal Form Game)

A normal form game consists of:

1. Players $N = \{1, 2, \ldots, n\}$
2. Strategy sets S_i for each player i
3. Payoff functions $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$

Game Theory: Strategic Decision Making

Theorem (Nash Equilibrium Existence)

Every finite game has at least one Nash equilibrium in mixed strategies.

Nash Equilibrium: Strategy profile where no player can unilaterally improve their payoff.

Game Theory: Strategic Decision Making

Why Game Theory Matters:

- Analyzes strategic interactions and competition
- Predicts outcomes in multi-agent systems
- Guides mechanism design and policy
- Foundation for understanding cooperation and conflict

Application: Cybersecurity Investment

Example (Cybersecurity Game)

Two firms choosing investment levels: High (H) or Low (L)
Payoff matrix:

Firm 1	H	L
H	(7, 7)	(5, 9)
L	(9, 5)	(3, 3)

Application: Cybersecurity Investment

Analysis:

- Dominant strategy: Choose L (low investment)
- Nash equilibrium: (L, L) with payoffs (3, 3)
- Prisoner's dilemma structure
- Both firms would benefit from coordination

Application: Cybersecurity Investment

Policy Implications:

- Market failure in cybersecurity investment
- Need for regulation or coordination mechanisms
- Information sharing reduces strategic uncertainty
- Demonstrates why security is often inadequate

Real-World Relevance

This model explains systematic under-investment in cybersecurity across industries.

Jackson Networks: Complex Stochastic Systems

Definition (Jackson Network)

A Jackson network consists of:

1. M service nodes (M/M/1 or M/M/∞ queues)
2. External arrival rates γ_i to node i
3. Routing probabilities p_{ij} from node i to j
4. Service rates μ_i at node i

Jackson Networks: Complex Stochastic Systems

Theorem (Jackson Network Product Form)

The steady-state distribution has product form:

$$P(n_1, \ldots, n_M) = \prod_{i=1}^{M} P_i(n_i)$$

where arrival rates satisfy:

$$\lambda_i = \gamma_i + \sum_{j=1}^{M} \lambda_j p_{ji}$$

Jackson Networks: Complex Stochastic Systems

Remarkable Implications:

- Complex networks decompose into independent queues
- Dramatically simplifies analysis of large systems
- Enables optimization of network performance
- Foundation for modern communication and logistics systems

Applications:

- Internet routing and traffic management
- Supply chain optimization
- Transportation network design

Integrating Probabilistic Models

What We've Accomplished:

- Markov chains for state-dependent evolution
- Queueing theory for service system optimization
- Reliability models for complex system design
- Decision theory for optimal choices under uncertainty
- Game theory for strategic interactions
- Stochastic networks for complex system analysis

Integrating Probabilistic Models

Mathematical Foundations:

- Rigorous probability theory ensures reliable analysis
- Matrix methods enable computational solutions
- Optimization theory guides decision-making
- Equilibrium concepts predict long-term behavior

Integrating Probabilistic Models

Future Directions:

- Integration with machine learning and AI
- Real-time optimization and control
- Network science and complex systems
- Behavioral economics and bounded rationality

Continuing Evolution

Discrete probabilistic modeling continues expanding into new domains as computational capabilities and mathematical techniques advance.

Practical Impact and Applications

Healthcare Systems:

- Emergency department optimization through queueing theory
- Epidemic modeling and intervention strategies
- Medical decision-making under diagnostic uncertainty
- Hospital resource allocation and capacity planning

Practical Impact and Applications

Technology and Engineering:

- Network reliability and communication protocol design
- Supply chain resilience and risk management
- Cybersecurity strategy and investment decisions
- System maintenance and lifecycle optimization

Practical Impact and Applications

Finance and Economics:

- Risk assessment and portfolio optimization
- Market microstructure and algorithmic trading
- Insurance pricing and actuarial modeling
- Strategic business decisions and competitive analysis

These applications demonstrate how mathematical rigor translates into practical value across diverse domains.

Key Takeaways

- Discrete probabilistic models capture essential uncertainty while maintaining tractability
- Mathematical rigor ensures reliable foundations for decision-making
- Integration of probability, optimization, and strategic thinking creates powerful frameworks
- Computational methods enable application to large-scale real-world problems
- Continued evolution expands capabilities for emerging challenges

The mathematical foundations established here provide reliable tools for navigating an increasingly uncertain and complex world.

Thank You

Questions and Discussion

Mathematical frameworks for uncertainty and strategic decisions

Next Chapter Preview:

Continuous Optimization Modeling

Building on probabilistic foundations for optimization under uncertainty